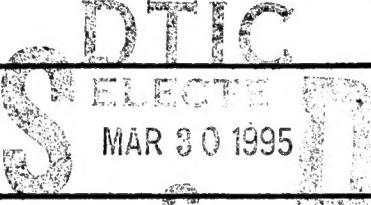
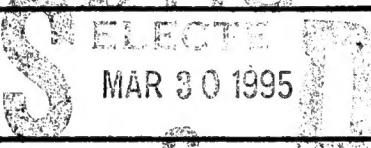


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13. ABSTRACT (Maximum: 200 words)		

The report discusses methods for the design of reconfigurable flight control systems. Model reference adaptive control algorithms were developed which allow for the rapid identification of aircraft parameters after failures or damages and for automatic control redesign. These algorithms are simple enough that they can be implemented in real-time with existing computers. Simulation results were obtained using a detailed nonlinear model of a fighter aircraft. The results demonstrate the ability of the adaptive algorithms to automatically readjust trim values after failures, to restore tracking of the pilot commands despite the loss of actuator effectiveness, and to coordinate the use of the remaining surfaces in order to maintain the decoupling between the rotational axes. The report also discusses analytical results that were obtained during the course of the research regarding the stability and convergence properties of multivariable adaptive control algorithms, the use of averaging methods for the analysis of transient response and robustness, and the estimation of uncertainty in dynamic models. A modified recursive least-squares algorithm with forgetting factor is also described which gives fast adaptation under normal conditions while keeping the sensitivity to noise limited when signal to noise ratios are poor.

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Final Report -- February 15, 1995
NEW ADAPTIVE METHODS FOR
RECONFIGURABLE FLIGHT CONTROL SYSTEMS

Grant: AFOSR F49620-92-J-0386

Period: July 1, 1992 to December 31, 1994

*Principal Investigator: Marc Bodson **

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1. Motivation and Objectives

Reconfiguration is likely to be a feature of future generations of flight control systems. The main motivation for reconfiguration is greater survivability, attained through the ability of the feedback system to reorganize itself in the presence of actuator failures and surface damage. High-performance aircrafts are often unstable to the point of exceeding the control capabilities of human pilots, either because of requirements of maneuverability, or to optimize efficiency. Instability will therefore become an increasingly common characteristic of fighter aircrafts and the need for fault tolerant control methods will become increasingly critical. In addition, the benefits of reconfigurable flight control systems extend beyond the immediate considerations of safety. Since reconfigurable systems reduce the need for other forms of reliability, such as redundant actuators, increased maintainability and reduced costs are expected to result from this technology.

In the past, the Air Force has supported a major R&D program through the *Self-Repairing Flight Control System (SRFCS)* program, administered through Wright-Patterson Air Force Base. A successful flight test marked the end of the first phase of the program. The approach followed in this program was based on the concept of failure detection and identification: a fast and efficient method to categorize the failure among a set of pre-planned conditions was combined with procedures to handle each of the conditions. In general, this approach worked well, but it presented significant problems. For a large number of possible failures, it is difficult and time-consuming to carry out the detection and classification. In the case of flight control, there are multiple possible actuator failures (multiple actuators and multiple failure modes, such as locked or floating) and an infinite variety of possible surface damages. In addition, a problem is that failure detection relies on models of the unfailed system. Therefore, any discrepancy between the model and reality can lead to false detection. The nonlinearity and the complexity of aircraft dynamics (especially engine dynamics and aerodynamics) make robust failure detection and classification a particularly difficult problem.

This report describes research results obtained for the design of reconfigurable flight control systems using adaptive methods. In this approach, the dynamic behavior of the aircraft is identified in real-time and a controller is designed automatically. Because failure classification is not relied upon, the resulting system is expected to tolerate a larger class of failures, including some that may

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not have been anticipated originally. Fundamental research problems that were addressed in the course of the research include the design of multivariable model reference adaptive control algorithms, and the improvement of their transient and robustness properties. Simulation results were obtained using a detailed nonlinear aircraft model. They demonstrate the ability of the adaptive algorithms to quickly readjust trim, to restore tracking of the pilot commands despite the loss of actuator effectiveness, and to coordinate the use of the remaining active control surfaces to maintain the decoupling between the rotational axes. The report also discusses methods for estimation of uncertainty in dynamic models and a modified least-squares algorithm with forgetting factor. This algorithm yields fast adaptation under normal conditions, but reduced sensitivity to noise when signal to noise ratios are poor.

2. Overview of Research Results

2.1 References

The research results are available in references [1] to [10]. These references acknowledge AFOSR support and are provided in Appendix. Further publications are expected to be derived from [1], [2], and [3]. We present a brief overview of these results.

2.2 Multivariable Algorithms for Reconfigurable Flight Control

Several problems concerning the design of multivariable adaptive control algorithms were investigated, with testing of these algorithms in simulations. Reconfigurable flight control requires multivariable strategies because strong cross-couplings usually appear after actuator failures. A Ph.D. thesis [10], initiated before the grant, was completed. In it, various aspects of multivariable adaptive control system design were addressed and the results appeared in conference and journal articles [4], [7], and [9]. In [4], the problem of parameter convergence and sufficient excitation was addressed. Necessary and sufficient conditions were given on the spectrum of reference inputs so that parameter convergence could be guaranteed. In the process, it was found that existing algorithms had to be modified to ensure convergence. Without such modifications, parameter convergence was not only impossible to guarantee, but, further, examples were found where instabilities occurred in the presence of unmodelled dynamics that could be tolerated with modified algorithms.

In [7], [9], the problem of model reference adaptive control for multivariable systems with unknown high-frequency gain matrices was considered. In flight control, the high-frequency gain matrix is the matrix giving the forces and moments provided by the actuators. This matrix is likely to change drastically after failures. Prior to our research, model reference adaptive control algorithms required this matrix to be positive definite, an assumption that would be difficult to justify in the context of reconfigurable flight control. A solution was developed that only required a bound on the norm of the high-frequency gain matrix to be known. A sort of hysteresis modification was used and the stability of the overall adaptive system was established analytically. Examples showed that the modification proposed in the paper was necessary to guarantee the stability of the adaptive system.

Special forms of multivariable model reference adaptive control algorithms were derived that were suitable for reconfigurable control [2]. Specifically, algorithms were developed that were applicable when full state information was available and when constant disturbances were present (the constant disturbances represented trim changes after failures). The control algorithm that was obtained had a nominal structure involving a constant feedforward gain matrix, a state feedback gain matrix, and a set of constant biases or trim values. This structure is highly similar to existing flight control

system laws. In this case however, the structure was augmented with a procedure to determine the parameters adaptively.

The multivariable adaptive control algorithms were evaluated in simulations [2], [3]. A detailed aircraft model was implemented, which was the model used for the *AIAA Design Challenge*. This model of a twin-engine fighter aircraft is a full nonlinear simulation that includes models of actuator dynamics (including limits and rate saturation), engine dynamics, aerodynamic forces, and atmospheric properties. The model also allows to study single actuator failures and the resulting cross-couplings between the longitudinal and lateral motions. Trim biases occurring after failures and dynamic variations with flight conditions are represented through the nonlinear model.

Using simulated data, the parameters of the linearized model of the aircraft were identified. A batch least-squares algorithm was applied to the responses of the original aircraft and of the aircraft with a locked left horizontal tail surface. The parameters of a model reference control law were calculated from the estimated aircraft parameters and were also independently estimated using a direct input error algorithm. Good agreement was found between the direct and indirect procedures, with the direct algorithm requiring somewhat less computations.

Very good tracking of pitch rate, roll rate and yaw rate commands was observed when the model reference control law was used. The responses of the control law designed for the unfailed aircraft were found to deteriorate significantly after the failure, but were restored if new controller parameters were used. This is illustrated in Figs. 1 and 2. In Fig. 1, the solid lines are the desired responses, specified by the reference model. The dashed lines are the aircraft responses. On the first row are the responses in pitch rate and roll rate to pitch rate commands. On the second row are the responses in pitch rate and roll rate to roll rate commands. In this simulation, the control law was the one designed for the original aircraft, although the actual aircraft had a locked left horizontal tail. The pitch rate response is seen to be significantly less than the desired response. The roll rate response is also smaller than specified, because roll control is partly achieved through the elevators in this aircraft. There is a significant cross-coupling from the pitch rate command to roll rate. This is due to the loss of symmetry in the elevator command and to the production of a large rolling moment.

The responses with the automatically redesigned controller are shown on Fig. 2. The dashed lines are the responses obtained with a fixed controller, and the dot-dashed lines with an adaptive controller. The fixed controller was obtained by taking data for the failed aircraft over a short period of time prior to the experiment. In the adaptive controller, the control law was determined and applied in real-time. The responses show that tracking of the commands was restored to the desired values. Cross-couplings which appeared after the failure (as roll responses to pitch rate commands) were reduced by a factor of 2 with the fixed controller, and to negligible values when an adaptive control law was used. Implicit in these responses is the correct identification of the trim values needed to maintain level flight. This property is important since not only the dynamic parameters of the aircraft may change after a failure, but the actuator positions required to avoid large transient responses may also be significantly different.

2.3 Robust Adaptive Algorithms

Research progressed towards improving the understanding of the robustness properties of adaptive control algorithms as they apply to reconfigurable flight control. In particular, the issues of transient performance and robustness in adaptive systems were studied. The paper [8] described a phenomenon similar to the burst phenomenon in adaptive systems without noise or unmodelled dynamics: despite theoretical results guaranteeing stability and convergence of tracking errors to

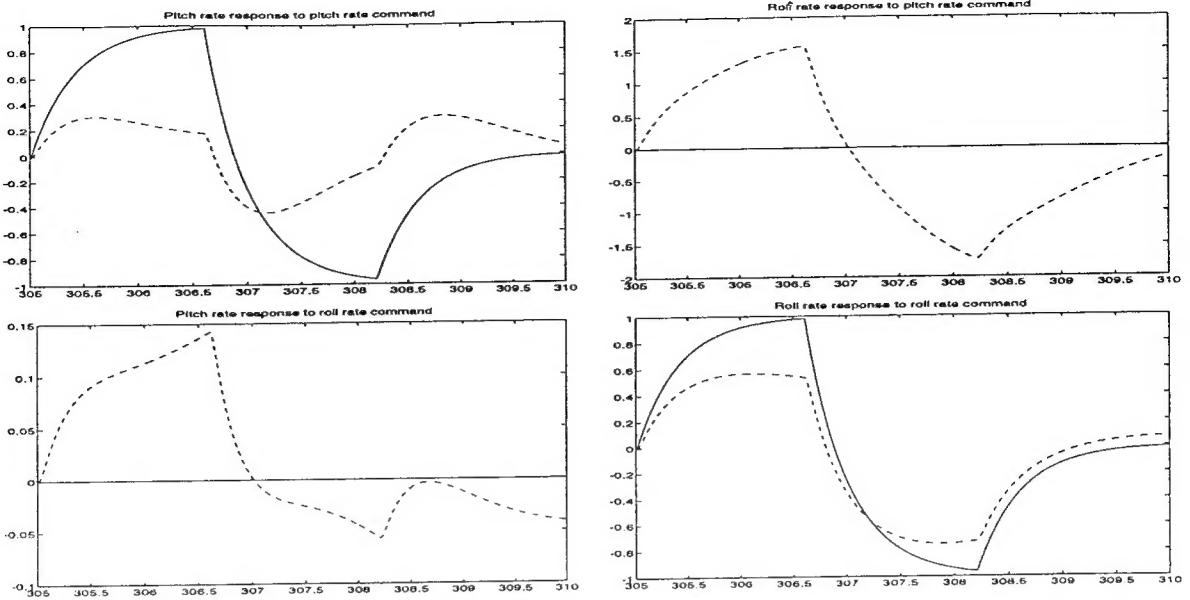


Figure 1: Aircraft Responses -- Original Control Law

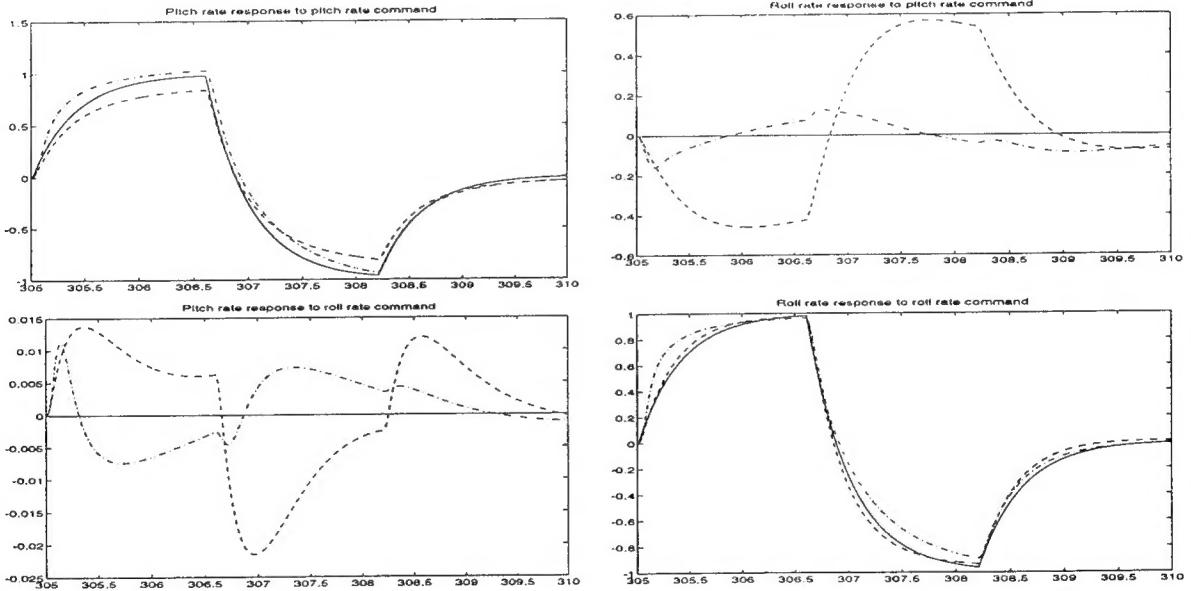


Figure 2: Aircraft Responses -- Redesigned Control Law

zero, the transient response of certain adaptive algorithms was found to exhibit large peaks even in ideal conditions. The problem was analyzed and was attributed to slow parameter convergence. The study demonstrated the critical importance of excitation in adaptive systems, and the need for fast and efficient adaptive algorithms (such as least-squares algorithms) in adaptive stabilization problems.

An instability mechanism was studied in [5], which was observed in an adaptive system excited by several sinusoids. In this case, there was no question of insufficient excitation, but the adaptive system was based on a reduced-order model. The instability arose even though the

adaptive system was stable when only one sinusoidal component was present. In the paper, the phenomenon was analyzed and explained using averaging methods. It was also shown that an input error algorithm remained stable and exhibited superior performance. The paper showed the importance of the proper choice of adaptive algorithm for robustness considerations. In general, it was found that algorithms with indirect adaptation structure, such as indirect adaptive control algorithms and so-called input error model reference adaptive control systems, were more robust in the presence of unmodelled dynamics.

Several methods were studied for the purpose of obtaining reliable estimates of uncertainty in the parameters identified through a least-squares algorithm. Estimates based on a stochastic analysis, an analysis assuming bounded noise, and a sensitivity analysis were compared in [6]. The error measures were tested using experimental data obtained on a DC motor. It was found that the stochastic estimates were unrealistically optimistic, because of the presence of correlated noise. Sensitivity-based estimates were found to more accurately reflect the uncertainty in the parameter estimates. The bounded noise analysis yielded similar results but required *a priori* estimates of the noise bounds. A linearized longitudinal model of an F-16, developed at Wright Patterson AFB, was also used in the study. The interest of this model was that it represented a modern, high-performance fighter aircraft with unstable dynamics. Fairly detailed models of sensor noise were also included. It was again found that the estimates of uncertainty provided by the sensitivity estimates were more reliable than the stochastic estimates.

2.4 Modified Least-Squares with Forgetting Factor

Thanks for Mr. Phil Chandler of Wright Patterson AFB, the PI became aware of the report [11] for the Phase I SBIR project: "Self-Designing Flight Control Using Modified Sequential Least-Squares Parameter Estimation and Optimal Receding Horizon Control Laws," supported by the AFOSR. In the report, a modified least-squares algorithm was presented and its advantages for flight control reconfiguration demonstrated. Although such an opportunity was not anticipated in the original proposal, the PI investigated the concept of the modified least-squares and obtained a recursive algorithm based on the same idea. An averaging analysis was carried out that showed that the algorithm exhibited some very interesting features, including:

- a variable data forgetting capability, so that the memory of the algorithm was long when the excitation was poor, and short when the excitation was significant;
- a second-order filtering capability, as opposed to the first-order filtering of the original algorithm, reducing the rate of fluctuations of the parameters;
- a stabilizing effect on the covariance matrix update, similar to some previously discussed in the literature.

The results were reported in [1]. Fig. 3 from the paper shows the main two features mentioned above. On the left is the response of the modified algorithm. On the right is the response of the standard least-squares algorithm with forgetting factor. The true parameter varies between 1 and -1 and appears as a triple step function. The smooth curves are the responses predicted by the averaging analysis, while the jagged curves are the responses of the parameter estimate. The approximate responses provided by the averaging analysis are found to be close to the original responses. At time 400, there is a drop in excitation (and therefore of the signal-to-noise ratio). The response of the modified algorithm is seen to slow down after that, while the parameter fluctuations remain limited. For the standard least-squares with (the same) forgetting factor, it is found that when the excitation drops, the fluctuations in the parameters become very large. These fluctuations could be

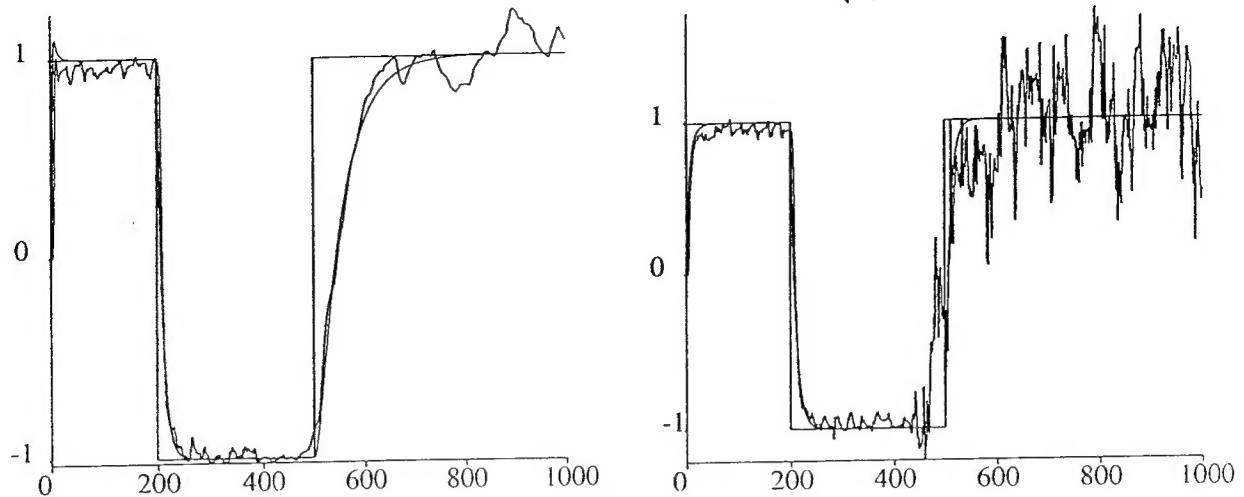


Figure 3: Adaptive Parameter Responses
 Left: Modified LS Right: Original LS

reduced by using a different forgetting factor, but the response in the first part of the simulation would then be slower. With the modified algorithm, a better tradeoff is obtained so that the algorithm reacts fast when there is sufficient excitation, yet keeps parameter fluctuations reasonable when the information available cannot sustain a fast adaptation.

3. Interactions with Air Force Labs

During the period of the grant, the PI interacted extensively with researchers at Wright Patterson Air Force Base involved in the reconfigurable flight control work, in particular Mr. Phil Chandler and Mr. Mark Mears of Wright Patterson AFB, and Dr. Meir Pachter of the Air Force Institute of Technology. The PI visited Wright Patterson AFB on the following occasions:

- An initial visit at the start of the grant (May 29, 1992). Shortly after that, Dr. Siva Banda visited the PI at Carnegie Mellon University (October 26, 1992). Dr. Banda gave a presentation on multivariable control methods for flight control, and visited the university facilities.
- The 2nd quarterly review meeting for the contract: "Application of Multivariable Control Theory to Aircraft Control Laws," (September 23, 1993). The PI later submitted a list of comments on the "zeroth draft" of the report [12] and used some of the concepts of the report in this project.
- The kick-off meeting for the SBIR Phase II effort "Self-Designing Controller" conducted by Barron Associates Inc., with the participation of Lockheed Fort Worth and Calspan Co (October 17, 1994).

Technical discussions with researchers of Wright Patterson AFB were also held while at conferences, including:

- the *Automatic Control Conference* (June 24-26, 1992)
- the *IEEE Conference on Control Applications* in Dayton, OH (September 14-16, 1992)
- the *IEEE Conference on Decision and Control* in Tucson, AZ (December 16-18, 1992)
- the *Eglin Guidance and Control Workshop* organized by Dr. Jim Cloutier of Eglin Air Force Base, in Fort Walton Beach, FL (May 12-14, 1993)

- the *Air Force Contractors Meeting* in Ann Arbor, MI (May 24-25, 1993)
- the *American Control Conference* in San Francisco, CA (June 2-4, 1993)
- the *IEEE Conference on Decision and Control*, in San Antonio, TX (December 15-17, 1993)
- the *Eglin Guidance and Control Workshop* organized by Dr. Jim Cloutier of Eglin Air Force Base, in Fort Walton Beach, FL (April 26-28, 1994)
- the *AFOSR Workshop on Dynamics and Control*, in Wright Patterson Air Force Base, OH (June 13-15, 1994).
- the *IEEE Conference on Decision and Control*, in Orlando, FL (December 14-16, 1994).

The PI organized an invited session titled: "Identification with Modelling Uncertainty and Reconfigurable Control" that was presented at the *IEEE Conference on Decision and Control*, in San Antonio, TX (December 15-17, 1993). The session was organized to bring together reputed researchers in the area of system identification and aerospace applications. A special emphasis of the session was on identification techniques that account for modelling uncertainty, and on the use of these methods for automatic control reconfiguration. The participants included: Prof. L. Ljung (Linkoping), Dr. V. Klein (NASA), Prof. G. Goodwin (Newcastle), Mr. P. Chandler (WPAFB), Dr. D. Bayard (JPL), and Prof. E. Wu (SUNY).

Together with Mr. Phil Chandler and Dr. M. Pachter, the PI also submitted an invited session titled "Nonlinear and Adaptive Methods for Reconfigurable Flight Control." The session was accepted for presentation at the *American Control Conference* to be held in Seattle, WA, on June 21-23, 1995. It was organized to give visibility to the work on reconfigurable control and to bring together researchers working in various organizations. In addition to the organizers, the participants of the invited session include: R. Barron (Barron Assoc.), Professor H. Michalska (Mc Gill University), H. Youssef (Lockheed Advanced Development Corporation), and A. Caglayan (Charles River Analytics).

4. Transitions

The PI has been in contact with Dr. D. Ward, of Barron Associates Inc., with an interest in implementing the recursive form of the modified least-squares algorithm as part of their SBIR Phase II effort. As mentioned earlier, the PI developed the recursive form of the algorithm, based on the concepts delineated in BAI's Phase I report. Since the kick-off meeting for the Phase II effort, the PI has had several discussions with D. Ward regarding the implementation of the algorithm and sent him Matlab files coding the algorithm in three forms, an exact form and two approximate but less computationally demanding forms.

The PI has also been interacting with Professor W. Messner, of Carnegie Mellon University, on a project for the control of tape drives systems. This project is carried out in cooperation with Datatape (a division of Kodak) and represents an interesting dual-use of the technology for multivariable adaptive systems. The need for adaptation arises from several sources including variations of tape characteristics with temperature, humidity, and tape manufacturer, and uncertainty in the tape's position.

5. Personnel

M. de Mathelin, graduate student, completed his Ph.D. thesis [10] under the grant in January 1993. J. Groszkiewicz, graduate student, carried out and completed his M.S. thesis [3] under the grant in December 1994.

K.-C. Choi, undergraduate student, worked under the grant during the Summer 1993.

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Appendix I

An Adaptive Algorithm with Information-Dependent Data Forgetting

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Abstract: An adaptive algorithm is derived, based on a weighted least-squares criterion incorporating a penalty on parameter variations. The result is a recursive algorithm whose covariance matrix update is identical to existing regularized or stabilized least-squares algorithms. In particular, the covariance matrix and its inverse are bounded under weak conditions. The parameter update equation is different from conventional algorithms and exhibits a second-order response, as opposed to the first-order response of the standard weighted least-squares algorithm. An averaging analysis shows that a linear time-invariant approximation of the parameter response gives useful information about the system. The approximate system has two sets of poles, with one set being the same as the one obtained for the weighted least-squares algorithm. The other set of poles is close to the origin in the z -plane when there is significant excitation, and close to the number one when the excitation vanishes. The results allow to select design parameters so that the algorithm reacts with a fast response under normal conditions, but slows down when the excitation is insufficient to yield adequate noise rejection. Simulations demonstrate on a simple example the properties obtained analytically.

1 Introduction

There is a well-known trade-off in adaptive control between the rate of convergence of the adaptive parameters and the sensitivity of their responses to noise. Rapid adaptation is particularly important in applications such as flight control reconfiguration for unstable aircrafts. However, the speed of response of an adaptive algorithm is limited in practice by the amount of noise and unmodelled dynamics that must be tolerated.

Least-squares algorithms are generally preferred over gradient algorithms because of their superior convergence properties. Exponential data weighting, *i.e.*, the forgetting factor approach, is the most common modification to the least-squares algorithm. Forgetting is incorporated to ensure that the algorithm is able to track parameter variations. Unfortunately, a problem is that the response of the algorithm is unbounded if the regressor vector is not persistently exciting. This problem has been previously addressed through covariance resetting [6], stabilized covariance updates [4] and variable forgetting factor approaches [2], [3].

In [8], a batch form of the least-squares algorithm with forgetting factor was used in a study of flight control reconfiguration. The authors faced an unacceptable trade-off: either the response of the algorithm was too slow, or the fluctuations due to the noise were too large. To solve the problem, they proposed to modify the algorithm so that a penalty on the parameter variations was included

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in the least-squares criterion. This allowed for the use of a smaller forgetting factor, yielding shorter memory and therefore faster response, yet with acceptable parameter fluctuations in the presence of noise. Simulations demonstrated this property, although no analysis was given.

In this paper, we use the same idea as proposed by [8], but instead derive a recursive algorithm. Because of differences in problem formulation, it is not clear that the recursive algorithm is equivalent to the batch algorithm studied in [8], although it is based on the same principle and exhibits similar properties. In addition to real-time implementation, an advantage of the recursive form is that it allows for comparison with the standard least-squares algorithm with forgetting factor and for a more detailed analysis.

The covariance matrix update of the new algorithm is the same as the Levenberg-Marquardt regularization method [5], also a special case of the stabilized least-squares algorithms of [4]. On the other hand, the adaptive parameter update is unconventional. The first-order difference equation is replaced by a second-order equation. Insight into the dynamic properties of the parameter update is obtained through the application of averaging methods [7], [1].

It is found, using the averaging approximation, that the covariance matrix equation is exponentially stable, with an equilibrium matrix that is bounded and positive definite. The parameter update equation is approximated asymptotically by a second-order system with two sets of poles. One set is the same as obtained with a comparable algorithm with forgetting factor but without the penalty on the parameter variations. The second set of poles varies, depending on the amount of excitation. When there is sufficient excitation, the second set of poles is close to the origin in the z -plane. Therefore, the convergence rate is determined by the first set of poles, *i.e.*, by the forgetting factor. When the excitation is reduced, the second set of poles moves towards the number one. Therefore, for sufficiently small values of the excitation, the convergence slows down and the effective memory is increased. Simulations are given to illustrate the results, showing that convergence is fast when there is significant excitation, while fluctuations due to noise remain acceptable when the excitation is poor.

2 Adaptive Algorithm

We consider the standard problem where, at discrete instants k , with $k = 1 \dots n$, measurements of a scalar signal $y(k)$, and of a so-called regressor vector $w(k)$ are obtained. In ideal conditions, these signals are assumed to satisfy the linear relationship

$$y(k) = w^T(k)\theta^* \quad (1)$$

where θ^* is a vector of unknown parameters, to be determined by the adaptive algorithm. In practice, equation (1) is affected by noise and other unmodelled effects. Further,

it is possible that the vector θ^* may vary slowly, or change abruptly.

At time n , we consider the following least-squares criterion

$$E[\theta(n)] = \sum_{k=1}^n (y(k) - w^T(k)\theta(n))^2 \lambda^{n-k} + \alpha |\theta(n) - \theta(n-1)|^2 \quad (2)$$

For $\alpha = 0$, this is the standard least-squares criterion with exponential data weighting, also called forgetting factor. The parameter λ is the forgetting factor, typically chosen between 0.9 and 0.99. For $\alpha \neq 0$, a weighting term is added that penalizes large variations between the new parameter estimate $\theta(n)$ and the previous estimate $\theta(n-1)$.

Setting $\partial E / \partial \theta(n) = 0$ yields

$$\theta(n) = \left(\sum_{k=1}^n w(k)w^T(k)\lambda^{n-k} + \alpha I \right)^{-1} \cdot \left(\sum_{k=1}^n w(k)y(k)\lambda^{n-k} + \alpha\theta(n-1) \right) \quad (3)$$

This is the "batch" form of the adaptive algorithm, although it is not a true batch estimate, because of the dependency of $\theta(n)$ on $\theta(n-1)$. It is natural to define

$$P(n) = \left(\sum_{k=1}^n w(k)w^T(k)\lambda^{n-k} + \alpha I \right)^{-1} \quad (4)$$

For consistency with standard terminology, we will call this matrix the covariance matrix of the adaptive algorithm. A recursive form for the inverse of the covariance matrix is deduced from (4) to be

$$P^{-1}(n) = \lambda P^{-1}(n-1) + w(n)w^T(n) + \alpha(1-\lambda)I \quad (5)$$

Given (4), the initial condition for the recursive equation is $P^{-1}(0) = \alpha I$. Now, consider that

$$P^{-1}(n)\theta(n) = \sum_{k=1}^n w(k)y(k)\lambda^{n-k} + \alpha\theta(n-1) \quad (6)$$

to deduce that

$$P^{-1}(n)\theta(n) = w(n)y(n) + \lambda (P^{-1}(n-1)\theta(n-1) - \alpha\theta(n-2)) + \alpha\theta(n-1) \quad (7)$$

After a few steps, one finds that a recursive formula for the parameter $\theta(n)$ is

$$\theta(n) = \theta(n-1) + P(n)w(n)(y(n) - w^T(n)\theta(n-1)) + \alpha\lambda P(n)(\theta(n-1) - \theta(n-2)) \quad (8)$$

Comments: 1) Equations (5) and (8) form the adaptive algorithm. It can be checked that, for $\alpha = 0$, these equations are the same as the usual update laws for the least-squares algorithm with forgetting factor. An update law for the matrix $P(n)$ can be obtained from the update for $P^{-1}(n)$ by the use of the matrix inversion lemma (see section 5). This update must be performed first, then the matrix $P(n)$ must be used to calculate $\theta(n)$. Alternatively, an update law for $\theta(n)$ can be derived that requires $P(n-1)$ instead of $P(n)$. The update equations where

$P(n)$ and $\theta(n)$ are expressed in terms of $P(n-1)$ and $\theta(n-1)$ are perhaps more familiar than the ones shown here, but they are equivalent.

2) For $\alpha \neq 0$, the covariance matrix update is the same as the Levenberg-Marquardt regularization method [5], pp. 364-365. Kreisselmeier [4] also studied various stabilized least-squares algorithms, of which this update is a special case. The adaptive algorithm derived here shares the advantages and the disadvantages of the stabilized algorithms. The main advantage is that both the matrix P and its inverse P^{-1} are bounded (see section 3). In other words, the algorithm does not suffer from the drawbacks of the least-squares algorithm and of the least-squares algorithm with forgetting factor, namely that P goes to zero in the first case when there is excitation (thereby halting adaptation) and goes to infinity in the second case when there is no excitation. On the other hand, a drawback is that the use of the matrix inversion lemma does not yield a simple matrix covariance update because of the matrix $\alpha(1-\lambda)I$ that is found in the right-hand side. We defer discussion of this issue until section 5.

3) The update law for $\theta(n)$ is significantly different from the update law in the original least-squares algorithm with forgetting factor. We observe the addition of a term proportional to $\theta(n-1) - \theta(n-2)$. This transforms the update law from a first-order difference equation into a second-order equation. Considerably more insight will be gained into the dynamic behavior of this equation using an averaging analysis in section 4.

4) Note that the parameter α appears in three places, that is, in both update laws and as initial condition for the covariance matrix update. The initial condition for the parameter vector is arbitrary.

3 General Properties of the Covariance Matrix Update Law

The update law for the covariance matrix (5) possesses desirable properties, under very general conditions. Since the algorithm is the same as the first stabilized least-squares algorithm proposed in [4] (case $N = 1$), the properties found there can be directly applied here. For convenience of notation, we define $R(n) = P^{-1}(n)$. Also, we allow for arbitrary initial conditions $R(0) = R^T(0)$ (normally $R(0)$ would be equal to αI).

Fact: (a) $R(n) = R^T(n) \forall n$

(b) $R(n) = \alpha I$ is an exponentially stable equilibrium of the unforced equation (5) (that is, with $w(n) = 0$)

(c) $R(0) \geq \alpha I \Rightarrow R(n) \geq \alpha I \forall n$

(d) $R(0) \geq \alpha I$ and $w(n)$ bounded $\Rightarrow R(n)$ bounded

Comment: see [4] for a proof. The proof can also be reconstructed easily, given the linearity of equation (5). The fact formally specifies the properties previously mentioned, among others that, under weak conditions, both the covariance matrix and its inverse are bounded.

4 Averaging Analysis

The update law for the adaptive parameter is different from conventional update laws, and is more difficult to analyze. To gain insight into the dynamic properties of the overall adaptive system, we carry out an averaging analysis. Averaging has been applied successfully to adaptive systems, including in the case where the dynamics are nonlinear (see [7] in the continuous-time case and [1] in the discrete-time case, for example). The assumptions required for the approximations to be valid are, in general, that the exogenous signals (that is, $y(n)$, $w(n)$) are stationary, and that the states of the adaptive system vary slowly compared to the exogenous signals. Accordingly,

we require that $w(n)$ be stationary, meaning that the limit

$$R_w(n) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=k_0}^{k_0+N} w(k)w^T(k+n) \quad (9)$$

exists for all n and uniformly with respect to k_0 . Of specific interest is the matrix $R_w(0)$, which is symmetric and positive semi-definite. It is positive definite if and only if the regressor vector w is persistently exciting. It is also the average of the matrix $w(n)w^T(n)$.

For convenience, we also assume that $y(n) = w^T(n)\theta^*$ (no noise), and define the parameter error vector

$$\varphi(n) = \theta(n) - \theta^* \quad (10)$$

Then, the update laws can be written as

$$P^{-1}(n) = \lambda P^{-1}(n-1) + w(n)w^T(n) + \alpha(1-\lambda)I \quad (11)$$

$$\varphi(n) = \varphi(n-1) - P(n)w(n)w^T(n)\varphi(n-1) + \alpha\lambda P(n)(\varphi(n-1) - \varphi(n-2)) \quad (12)$$

The system is nonlinear and time-varying, with the time variation originating from the matrix of signals $w(n)w^T(n)$. To justify the averaging approximation (slow variation), we define $v(n)$ and σ through

$$w(n) = \sqrt{\varepsilon} v(n) \quad \lambda = 1 - \varepsilon\sigma \quad (13)$$

Then, (11) can be written as

$$P^{-1}(n) = P^{-1}(n-1) + \varepsilon(v(n)v^T(n) - \sigma P^{-1}(n-1) + \alpha\sigma I) \quad (14)$$

For ε sufficiently small, the response of this system can be approximated [1] by that of the averaged system

$$P_{av}^{-1}(n) = P_{av}^{-1}(n-1) + \varepsilon(R_v(0) - \sigma P_{av}^{-1}(n-1) + \alpha\sigma I) \quad (15)$$

In terms of the original variables,

$$P_{av}^{-1}(n) = \lambda P_{av}^{-1}(n-1) + R_w(0) + \alpha(1-\lambda)I \quad (16)$$

In summary, we found that, assuming that the regressor vector $w(n)$ is stationary and sufficiently small, and assuming that the forgetting factor λ is sufficiently close to one, the trajectories of the system (5) are arbitrarily close to those of the averaged system (16).

The equation for P_{av}^{-1} is a linear time-invariant system, which is exponentially stable. All the poles for this system are located at $z = \lambda$. The equilibrium matrix of the system is

$$(P^*)^{-1} = \frac{1}{1-\lambda}(R_w(0) + \alpha(1-\lambda)I) \quad (17)$$

Consistently with the results of section 3, the matrix $(P^*)^{-1}$ is bounded if w is bounded, and its inverse P^* is also bounded with $P^* \leq \frac{1}{\alpha}I$.

For the parameter response, averaging theory does not seem to be directly applicable. However, note that (11) and (12) lead to the expression

$$\varphi(n) = P(n)(\lambda P^{-1}(n-1)\varphi(n-1) + \alpha\varphi(n-1) - \alpha\varphi(n-2)) \quad (18)$$

This equation defines $\varphi(n)$ as a function of $P(n)$ alone, that is, without any direct input from $w(n)$. In this case,

we will define the averaged system as the system obtained by replacing $P(n)$ by $P_{av}(n)$ in (18). Using (16),

$$\varphi_{av}(n) = \varphi_{av}(n-1) - P_{av}(n)R_w(0)\varphi_{av}(n-1) + \alpha\lambda P_{av}(n)(\varphi_{av}(n-1) - \varphi_{av}(n-2)) \quad (19)$$

This expression leads to useful insight into the dynamics of the adaptive system. In section 6, we will also demonstrate through examples the closeness of this approximation.

Assuming that the matrix P_{av} has converged to its steady-state value, and noting that $P^*R_w(0) = (1-\lambda)I - \alpha(1-\lambda)P^*$, the dynamics of the parameter update law are given by

$$\varphi_{av}(n) = (\lambda I + \alpha P^*)\varphi_{av}(n-1) - \alpha\lambda P^*\varphi_{av}(n-2) \quad (20)$$

This is a linear time-invariant system. Its poles are given by the roots of

$$\det((z - \lambda I)(z - \alpha P^*)) = 0 \quad (21)$$

If the dimension of the parameter and regressor vectors is m , there will be $2m$ poles, with m of them located at $z = \lambda$ (the forgetting factor). The remaining m poles are located at the eigenvalues of αP^* . However,

$$\alpha P^* = \alpha(1-\lambda)(R_w(0) + \alpha(1-\lambda)I)^{-1} \quad (22)$$

If there is negligible excitation in all directions of the parameter space, the matrix $R_w(0)$ will be small, leading to a matrix αP^* close to the identity and eigenvalues close to 1. For large excitation, the matrix $R_w(0)$ will dominate, leading to a matrix αP^* close to zero and eigenvalues close to 0. In other words, the convergence will be slow and the memory long when there is little excitation. But fast responses and short memory will be achieved if the excitation is sufficiently large. Above a certain excitation level, however, the convergence rate does not increase and is determined by the other poles at $z = \lambda$ and the forgetting factor λ .

The threshold of excitation can be adjusted through proper choice of the design parameter α . In the scalar case, for example, the additional pole is located at λ if the power in the signal w is such that $R_w(0) = \frac{\alpha}{\lambda}(1-\lambda)^2$. For higher excitation levels, the additional pole will be faster than the pole at λ . For lower excitation levels, its response will be slower and will dominate. In the multivariable case, a similar result holds. Further, the variable forgetting capability is directional. For a two-dimensional system with significant excitation in one direction but poor excitation in the other, one pole will be close to zero and the other close to one (in addition to the two poles at λ). Extended memory will be obtained in the direction of poor excitation.

5 Implementation Issues

One of the difficulties with the implementation of the algorithm is that the update law is expressed in terms of $P^{-1}(n)$. In the original algorithm (with $\alpha = 0$), one may use the matrix inversion lemma

$$A^{-1} = B^{-1} + C C^T \quad (23)$$

$$\Rightarrow A = B - BC(I + C^T B C)^{-1}C^T B \quad (24)$$

to obtain a recursive formula for $P(n)$. Because the matrix $C = w$ is a vector, the matrix $I + C^T B C$ is a scalar and is easily inverted. In the case of the modified algorithm (5)-(8), the matrix inversion lemma could still be used, but the matrix C would have to be defined by

$$C = \begin{pmatrix} w & \sqrt{\alpha(1-\lambda)} I \end{pmatrix} \quad (25)$$

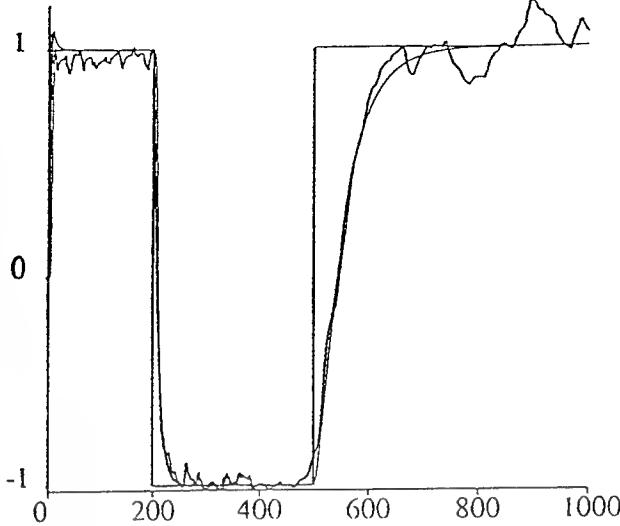


Figure 1: Parameter Response – Modified LS

so that the inversion of a matrix with dimension higher than $P(n)$ would be required. One might as well update $P^{-1}(n)$ and invert it to update $\theta(n)$ in that case. This problem is well-known and shared by the stabilized [4] or regularized [5] algorithms. A solution proposed by [5] consists in replacing $\alpha(1 - \lambda)I$ in the update law by $m\alpha(1 - \lambda)e_i e_i^T$, where e_i is a vector which is zero everywhere except at the i th position where it is one. m is the dimension of w . As time progresses, the index i is incremented and returned to one when the end of the vector is reached. With this modification, the matrix C becomes

$$C = \begin{pmatrix} w & \sqrt{\alpha(1 - \lambda)}e_i \end{pmatrix} \quad (26)$$

Through the use of the matrix inversion lemma, only a two-dimensional matrix needs to be inverted at each time step. Surprisingly, this modification induces no change to the averaged system definition, since the matrix $m\alpha(1 - \lambda)e_i e_i^T$ is stationary and has average $\alpha(1 - \lambda)I$, i.e., the same as the original matrix.

6 Example and Simulation Results

We consider the scalar example

$$y(k) = w(k)\theta^* + n(k) \quad (27)$$

where $n(k)$ is an additive noise consisting of independent, identically distributed gaussian random variables with zero mean and standard deviation 0.1. In the first part of the simulations

$$w(k) = \sin(2\pi k/50) \quad (28)$$

that is, $w(k)$ is a sinusoidal signal with magnitude 1 and period equal to 50 time steps. In the algorithm, the forgetting factor is set to $\lambda = 0.9$. This corresponds to a memory of 10 steps (if we use a definition equivalent to the continuous-time definition of time constant, i.e., $\lambda^{10} \simeq e^{-1}$). The parameter α is set to 2.45, for reasons to be explained later. In Figure 1, the response of the adaptive parameter is shown. The true parameter θ^* was equal to 1 from 0 to 200, -1 from 200 to 500, and 1 from 500 to 1000. It is shown on the figure as a discontinuous signal. Two additional responses are shown: the adaptive parameter response and the averaged adaptive parameter response, which is the smoother curve of the two. The proximity of these two responses

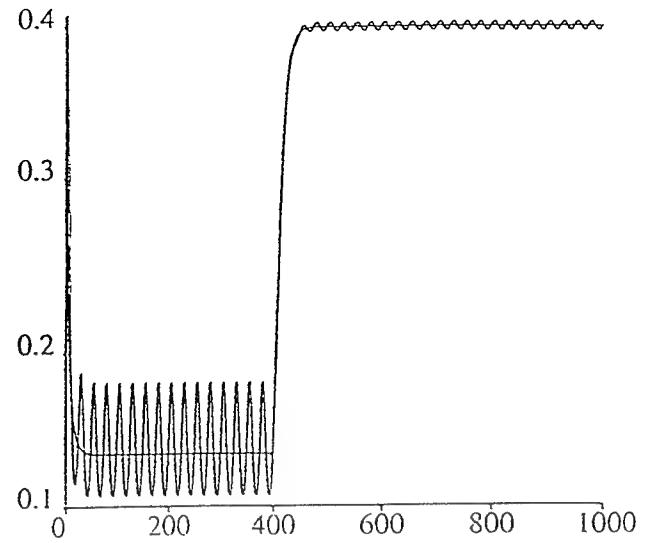


Figure 2: Covariance Matrix Response – Modified LS

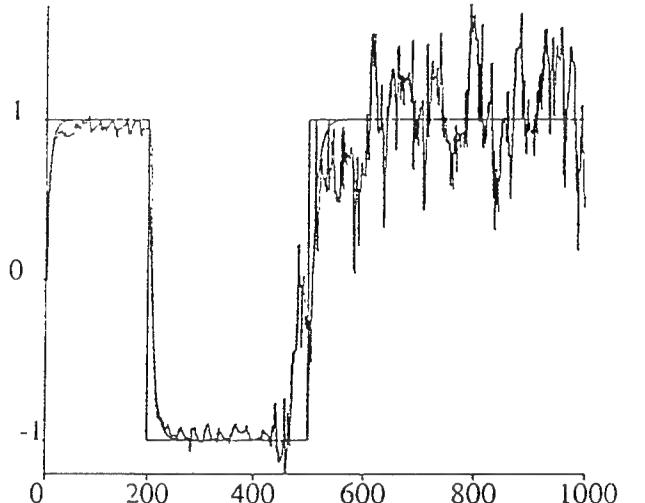


Figure 3: Parameter Response – Original LS

can be observed at once, demonstrating the value of the averaging approximation. At time 400, the magnitude of the signal $w(k)$ was reduced from 1 to 0.1. As a consequence, $R_w(0)$ was reduced from 0.5 to 0.005. The value of α is such that, for $R_w(0) = 0.005$, the second pole at $\alpha(1 - \lambda)/(\alpha(1 - \lambda) + R_w(0))$ is equal to 0.98, or a time constant of 50 steps. On the other hand, for $R_w(0) = 0.5$, the second pole is located at 0.33 and is therefore much faster than the pole at $\lambda = 0.9$. The threshold of excitation such that both poles are located at $\lambda = 0.9$ is $R_w(0) = 0.027$. The responses exhibit precisely the expected convergence rates, with a time constant of approximately 10 steps at $t = 200$ and 50 steps at $t = 500$.

Figure 2 shows the responses for the covariance matrix and its averaged equivalent (both matrices are scalars here). Again, the averaging approximation is a good predictor of the actual system's response. When the excitation diminishes at $t = 400$, the covariance matrix increases, but only up to approximately 0.4, which is close to the upper bound equal to $1/\alpha = 1/2.45$.

Figures 3 and 4 give the responses for $\alpha = 0$. This is the usual least-squares algorithm with forgetting factor.

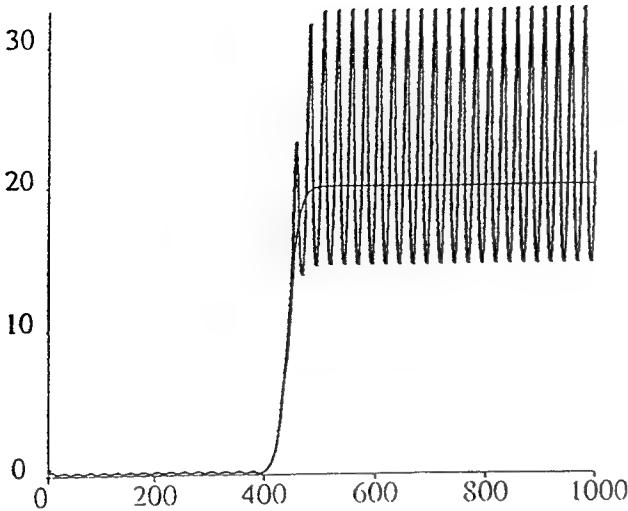


Figure 4: Covariance Matrix Response – Original LS

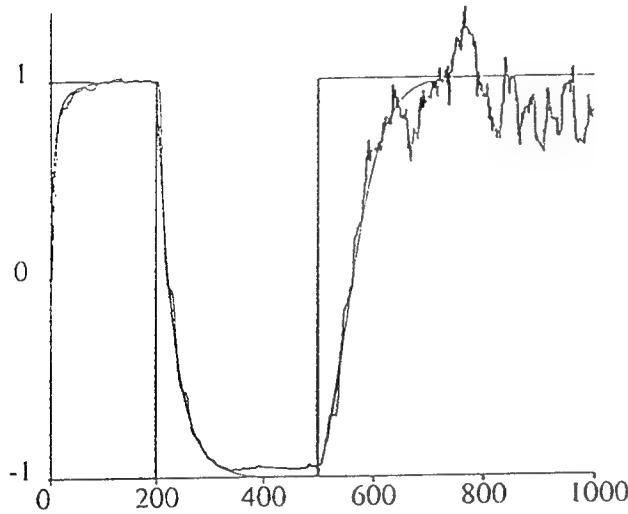


Figure 5: Parameter Response – Original LS

One finds that, once excitation diminishes, the parameter variations due to the noise become very large. This is due, in part, to the large values of the P matrix reaches in that case. These large parameter fluctuations are avoided in the modified algorithm because of the limits placed on the matrix P through the parameter α . In addition, increased filtering is provided in the modified algorithm through the change from a first-order update law to a second-order update law.

Figures 5 and 6 shows the response of the least-squares algorithm with forgetting factor $\lambda = 0.97$. This choice of forgetting factor leads to a smoothing of the parameter fluctuations in the second part of the response, when the excitation is reduced. However, in the earlier part of the response around $t = 200$, the convergence time is longer than for the modified algorithm. In the original algorithm, there is little flexibility to deal with varying excitation conditions. A single choice of forgetting factor must satisfy all conditions. The modified algorithm gives more flexibility and relaxes certain trade-offs. Note that, in Figure 5, the fluctuations of the parameter responses appear to be of higher frequency than in Figure 1. This

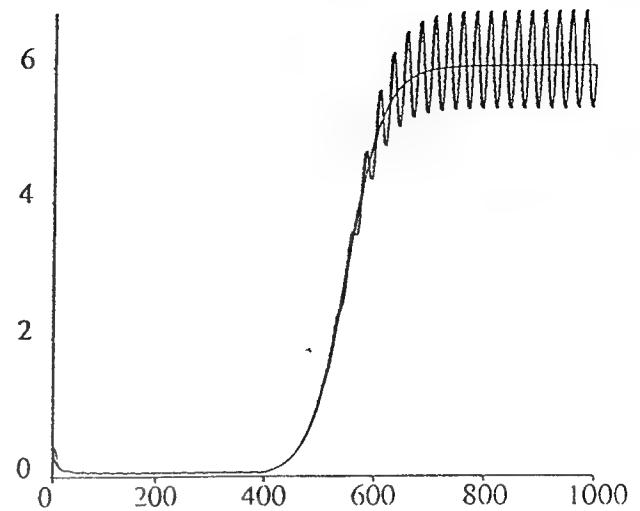


Figure 6: Covariance Matrix Response – Original LS

is again attributed to the difference between a first-order and second-order update law.

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Multivariable Adaptive Algorithms for Reconfigurable Flight Control

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Abstract: The application of multivariable adaptive control techniques to flight control reconfiguration is considered. The paper first discusses three adaptation mechanisms for model reference control. It is shown that simple algorithms can be obtained if full state feedback is assumed. The respective advantages and disadvantages of the three algorithms are discussed in general terms, considering their complexity and the assumptions that they require. Next, the application of the adaptive algorithms to reconfigurable flight control is investigated. The objective is to design automatically a flight control law in the presence of actuator failures or surface damage. Design considerations for the adaptive algorithms are discussed in this context. Simulations obtained using a full nonlinear simulation of a twin-engine jet aircraft are included to illustrate the results.

1 Introduction

Reconfiguration is likely to be a feature of future generations of flight control systems. The main motivation for reconfiguration is greater survivability, attained through the ability of the feedback system to reorganize itself in the presence of actuator failures and surface damage. High-performance aircrafts are often unstable to the point of exceeding the control capabilities of a human pilot. Instability follows either from requirements of maneuverability, or from efficiency considerations, and it is expected to become an increasingly common characteristic of future aircrafts. The need for fault tolerant control methods is therefore critical. The benefits of reconfigurable flight control systems extend beyond the immediate considerations of safety. Indeed, reconfigurable systems reduce the need for other forms of reliability, such as redundant actuators. Therefore, increased maintainability and reduced costs are expected to result from this technology [3].

In the past, the Air Force has supported a major R&D program through the *Self-Repairing Flight Control System (SRFCS)* program, administered through Wright-Patterson Air Force Base. A successful flight test marked the end of the first phase of this program [10]. The approach followed in this program was based on the concept of failure detection and identification. The system consisted in a fast and efficient method to detect the failure among a set of pre-planned conditions, and in procedures to handle each of the cases. This approach works well in restricted cases, but suffers from significant drawbacks. The first is that, as the number of failures grows, it becomes increasingly difficult and time-consuming to carry out the detection and classification. Further, there is no reason to believe that a failure that has not been categorized will not cause the whole system to fail. In the case of flight control, there are multiple possible actuator failures

(multiple actuators and multiple failure modes, such as locked or floating) and an infinite variety of possible surface damages. In addition, because failure detection relies on models of the unfailed system, any discrepancy between the model and reality can lead to false detection. Because of the nonlinearity and complexity of aircraft dynamics (especially engine dynamics and aerodynamics), this is a nontrivial problem in reconfigurable flight control.

A totally different approach to the problem of flight control reconfiguration consists in identifying the dynamic behavior of the aircraft in real-time, and in designing a controller automatically. Because such an approach does not rely on failure classification, it is expected that the resulting system will tolerate a larger class of failures, including some that may not have been anticipated. In this paper, we discuss several multivariable adaptive control algorithms that may be used with that objective in mind. We make assumptions that are realistic in the flight control problem, yet allow to considerably simplify the algorithms available in the literature. We also present the results of a simulation study using a detailed nonlinear model of a twin-engine aircraft.

2 Reconfigurable Flight Control

2.1 Aircraft Model

The dynamics of aircrafts can be accurately represented by nonlinear differential equations models. For the purpose of flight control, linearized models are typically used, of the form

$$\dot{x} = Ax + Bu \quad y = Cx \quad (1)$$

The components of the state x are divided into the longitudinal variables: α (angle of attack), q (pitch rate), h (altitude), v (velocity), and the lateral variables: β (sideslip), p (roll rate), r (yaw rate), ϕ (roll angle), ψ (yaw angle). The control inputs are also separated into the longitudinal control variables: δ_E (elevator command) and δ_T (thrust command), and the lateral control variables: δ_A (aileron command) and δ_R (rudder command). For the design of stability augmentation systems, the state vector can be reduced to the five states α , q , β , p , and r with good approximation. The fast control variables are also reduced to δ_E , δ_A , δ_R . We will use such a model in this paper (except for simulations).

The control outputs y can be defined in several ways. For low dynamic pressure and limited angle of attack, the variables q , p , r are good choices of output variables to be regulated (cf. [7]). The control problem is then that of a linear-time invariant system with three inputs and three outputs. Because of the symmetry of the (unfailed) aircraft, the motions in the longitudinal and lateral axes can be decoupled in general, so that the problem is actually reduced to a single-input single-output control problem in the longitudinal axis, and a 2×2 multivariable problem in the lateral axis.

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2.2 Effects of Failures

After a failure, a model similar to (1) can still be used for control system design. There are, however, two differences. First, the dynamics are generally not decoupled after a failure because symmetry is lost (both ailerons will not fail together). In addition, a failure is likely to alter the equilibrium of the aircraft. The constant values added to the controls (trim biases) then have to be modified to reach a new equilibrium. We account for this equilibrium change by introducing a state-space model

$$\dot{x} = Ax + Bu + d \quad y = Cx \quad (2)$$

where the constant disturbance d represents the bias forces introduced by the failure or damage. The lack of symmetry implies that the matrices A and B will not necessarily be block-diagonal anymore.

2.3 Challenges of Control Reconfiguration

There are several challenges in solving the flight control reconfiguration problem:

- it is a multivariable problem, with strong cross-couplings appearing after failures;
- it is a problem suited for adaptation, not only because the dynamic model after failure is unknown but also changes with flight condition;
- modern high-performance aircrafts are unstable, often leaving little time for reconfiguration;
- actuator authority is limited and sensor noise is significant (high-gain feedback is not an option);
- the reconfiguration strategy must be robust to linear and nonlinear unmodelled effects;
- the number of possible failures is large.

3 Multivariable Adaptive Algorithms

In this section, we discuss some adaptive algorithms that are relevant to the problem of flight control reconfiguration. The control objective that we consider is based on the well-known model reference formulation. Multivariable model reference adaptive control algorithms are available in the literature (see [8], [9], and the references therein). In the present paper, we discuss slightly different algorithms. We incorporate the disturbance d and we also show that simplifications can be obtained under the assumption of full state measurement. In that case, the "observer" part of the controllers can be eliminated and a state-feedback control law can be obtained.

3.1 Assumptions and Reference Model

We consider the state-space model for the plant (2), where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^m$, and $d \in \mathbb{R}^n$. We assume that the whole state x is available for measurement, although only the output y is to be tracked. The objective is for y to match the output y_M of a reference model

$$\dot{y}_M = A_M y_M + B_M r \quad (3)$$

where $y_M \in \mathbb{R}^m$ and $r \in \mathbb{R}^m$. The matrices A_M and B_M are arbitrary square matrices with A_M stable. For the model reference control problem to have a relatively simple solution, we assume:

Assumption 1 The plant has relative degree 1, i.e., $\det(CB) \neq 0$.

Assumption 2 The plant transfer function is minimum phase, i.e., the zeros of transmission of the system are in the open left-half plane.

The first assumption guarantees that the closed-loop transfer function of the plant can be made to match the transfer function of the reference model (3) using a proper compensator. If the assumption is not satisfied, the model reference control problem may still be solvable, but a more complex reference model would have to be chosen, so as to match the so-called *Hermite normal form* of the plant.

When Assumption 1 is satisfied, this Hermite form is simply $H(s) = \text{diag}\{1/s\}$ which means that the behavior of the plant at infinity is that of a multivariable integrator. The matrix CB is called the *high-frequency gain matrix* of the plant and is usually denoted K_P in the adaptive control literature. It is a most critical parameter for adaptive control algorithms.

The second assumption is a necessary assumption to guarantee the internal stability of the model reference control algorithm. The dimension of the state-space for the reference model is m , while the dimension of the state-space for the plant is n . Therefore, $n - m$ modes must be made unobservable or uncontrollable. It can be shown that the model reference control law places m modes of the plant at the desired model reference locations, and makes the others unobservable by placing them at the locations of the transmission zeros.

3.2 Model Reference Control Law

We consider the state-feedback control law

$$u = C_0 r + G_0 x + v \quad (4)$$

where $C_0 \in \mathbb{R}^{m \times m}$, $G_0 \in \mathbb{R}^{m \times n}$, $v \in \mathbb{R}^m$ are free controller parameters. The closed-loop dynamics are given by

$$\dot{x} = Ax + BC_0 r + BG_0 x + Bu + d \quad y = Cx \quad (5)$$

or, in terms of the output y ,

$$\dot{y} = (CA + CBG_0)x + CBC_0 r + CBv + Cd \quad (6)$$

(6) leads to the same input/output relationship as that of the reference model (3) for the so-called *nominal* values of the controller parameters

$$\begin{aligned} C_0^* &= (CB)^{-1} B_M \\ G_0^* &= (CB)^{-1} (A_M C - CA) \\ v^* &= -(CB)^{-1} (Cd) \end{aligned} \quad (7)$$

If the plant was known, these would be the controller parameters that one would use to achieve the model reference control objective.

3.3 Indirect Adaptive Control

An indirect adaptive controller can easily be derived using (7). Let \hat{A} , \hat{B} , \hat{d} be estimates of the plant parameters A , B , d . Then, an error can be defined by

$$e_1 = \hat{A}x + \hat{B}u + \hat{d} - \dot{x} \quad (8)$$

where e_1 is an error vector which, under the assumptions, is equal to

$$e_1 = (\hat{A} - A)x - (\hat{B} - B)u + (\hat{d} - d) \quad (9)$$

In practice, a least-squares algorithm can be used to find the estimates of A , B , and d which minimize the sum of squares of e_1 evaluated at several sampling instants. Each row of (8) is treated independently in this process. Such an algorithm requires the measurement of the derivative of x . In practice, the derivative can often be obtained by filtered differentiation of x (see [1] for example). In the context of flight control, such differentiation is not necessary because of the availability of direct acceleration measurements.

The complete adaptive control algorithm is obtained by setting the controller parameters C_0 , G_0 , and d in (4) using (7), with the estimates of the plant parameters \hat{A} , \hat{B} , \hat{d} replacing the true plant parameters A , B , and d . Aside from the issue of stability, a major question to be resolved is what to do when $C\hat{B}$ is singular, since this matrix is to be inverted. Few methods are available to address this problem satisfactorily, and it is still a subject of current research.

3.4 Direct Adaptive Control – Output Error

A totally different algorithm can be obtained, based on known properties of gradient algorithms (cf. [8]). The so-called *output error* $e_0 = y - y_M$ is used for that purpose. Because the system is assumed to have relative degree 1, a simple algorithm can be derived (there is no need for the so-called augmented error). We use the following facts.

Fact 1 The output error e_0 satisfies

$$e_0 = (sI - A_M)^{-1} (CB) [(C_0 - C_0^*) r + (G_0 - G_0^*) x + (v - v^*)] \quad (10)$$

Proof: Define $\dot{z}_M = A_M z_M + B G_0^* x_M + B v^* + d$, so that $y_M = C z_M$ for some appropriate choice of initial conditions for z_M . Since $\dot{e}_0 = C \dot{z} - \dot{y}_M = C \dot{z} - A_M C z_M - b_M r$, we have, using (7)

$$\dot{e}_0 = (CA + (CB) G_0^*) (x - x_M) + (CB) [(C_0 - C_0^*) r + (G_0 - G_0^*) x + (v - v^*)] \quad (11)$$

which leads to (10).

A compact form of equation (10) is

$$e_0 = (sI - A_M)^{-1} (CB) [\Phi \cdot w] \quad (12)$$

where the matrix

$$\Phi = ((C_0 - C_0^*), (G_0 - G_0^*), (v - v^*)) \quad (13)$$

is the $m \times (m + n + 1)$ matrix of parameter errors, and the vector $w^T = (r^T, x^T, 1)$ is the so-called *regressor vector* of dimension $m + n + 1$. The product $\Phi \cdot w$ is carried out in the time domain, and the resulting signal is applied to the linear time-invariant operator $(sI - A_M)^{-1} (CB)$.

Fact 2 The update law $\dot{\Phi} = -G_0 w^T$ leads to an adaptive system that is stable in the sense of Lyapunov, with the property that e_0 tends to zero as $t \rightarrow \infty$, provided that $(sI - A_M)^{-1}$ is a strictly positive real transfer function matrix, and $(CB)^T G^{-1}$ is a positive definite matrix.

Proof: follows from the Kalman-Yacubovich-Popov lemma. See, e.g., [8].

The algorithm given here is similar to the one available in [8], although with some nontrivial adjustments. The main differences are the simplifications resulting from state measurements and the constant disturbance. The SPR condition can be satisfied by choosing $A_M + A_M^T$ negative definite. The other condition requires prior knowledge of the matrix CB , which is the high-frequency matrix mentioned earlier, and a choice of the adaptation gain matrix G to enforce the condition.

3.5 Direct Adaptive Control – Input Error

Another direct adaptive algorithm can be obtained through the use of the following fact.

Fact 3 The following identity is satisfied for all time

$$u = C_0 B_M^{-1} (\dot{y} - A_M y) + G_0^* x + v^* \quad (14)$$

Proof: since $\dot{y} - A_M y = (CA - A_M C) x + (CB) u + Cd$, using the nominal parameter values (7), we find

$$\dot{y} - A_M y = (CB) (-G_0^* x + u - v^*) \quad (15)$$

and (14) follows.

The identity (14) leads to a new error equation

$$e_2 = C_0 B_M^{-1} (\dot{y} - A_M y) + G_0 x + v - u \quad (16)$$

Under the assumptions, we have that

$$e_2 = \Phi \cdot z \quad (17)$$

where Φ is the (controller) parameter error defined in (13) and z is a new regressor vector defined to be

$$z = \begin{pmatrix} B_M^{-1} (\dot{y} - A_M y) \\ x \\ 1 \end{pmatrix} \quad (18)$$

The main difference between the error equation (17) and (8) is the absence of the transfer function between the parameter error and the error signal, eliminating conditions necessary for the stability of the adaptive algorithm. Further, it makes possible the use of least-squares algorithms, either in batch or recursive forms.

$a =$

$$\begin{array}{cccccc} -0.7685 & 1.0137 & -0.0185 & 0.0019 & 0.0018 \\ -4.3448 & -1.9816 & 0.4991 & 0.0598 & 0.0788 \\ 0.2155 & -0.1958 & -0.0636 & 0.0585 & -0.9273 \\ -1.8760 & -0.4775 & -20.3609 & -1.3178 & 1.9133 \\ -0.0432 & 0.0018 & 4.9747 & -0.0017 & -0.4948 \end{array}$$

$b =$

$$\begin{array}{cccc} 0.0096 & -0.0193 & -0.0457 \\ -6.8978 & -0.3138 & -0.0961 \\ -0.2652 & -0.1649 & -0.0714 \\ -0.0131 & 18.7269 & -2.0886 \\ 0.0556 & 1.4760 & -2.6271 \end{array}$$

$d =$

$$\begin{array}{c} 3.5046 \\ -0.0148 \\ -1.6729 \\ 7.7000 \\ 0.1543 \end{array}$$

$c_0 =$

$$\begin{array}{ccc} -0.3623 & -0.0076 & 0.0193 \\ -0.0012 & 0.1424 & -0.1132 \\ -0.0083 & 0.0798 & -1.0148 \end{array}$$

$g_0 =$

$$\begin{array}{cccccc} -0.6350 & 0.0737 & -0.0279 & 0.0123 & 0.0018 \\ 0.1028 & 0.0275 & 1.3852 & -0.0674 & -0.0182 \\ 0.0279 & 0.0177 & 2.6712 & -0.0382 & 0.7531 \end{array}$$

$v =$

$$\begin{array}{c} 0.0200 \\ -0.4316 \\ -0.1833 \end{array}$$

Figure 1: Parameters for the Unfailed Aircraft

3.6 Comparison of Adaptive Algorithms

The three algorithms presented here all achieve the same model reference control objective, although through quite different structures and making somewhat different assumptions. The following elements are worth considering:

- Number of parameters:** both direct methods adjust the same number of parameters, which are the elements of C_0 , G_0 , and v . The total number is: $m^2 + m \cdot n + m$. In the case of the indirect algorithm, the parameters that are identified are the plant parameters, and there are $n^2 + n \cdot m + n$ of them. Since $m < n$, the direct methods require less parameters. For the case of aircraft control with a standard reduced-order model ($n = 5$, $m = 3$), the number of parameters is 45 in the case of the indirect algorithm, and 27 in the direct case, a substantial reduction.

- Prior information:** the direct output error algorithm requires the most restrictive assumptions, by imposing positive definiteness conditions on the product of the inverse of the adaptation gain matrix with the high-frequency gain matrix. Since there is no obvious way to enforce such a condition, the condition practically becomes a symmetry and positive definiteness condition on the high-frequency gain matrix itself. Guaranteeing that such a condition is satisfied is all but trivial. The direct input error and indirect algorithms also require conditions on the high-frequency gain matrix to be implementable, but

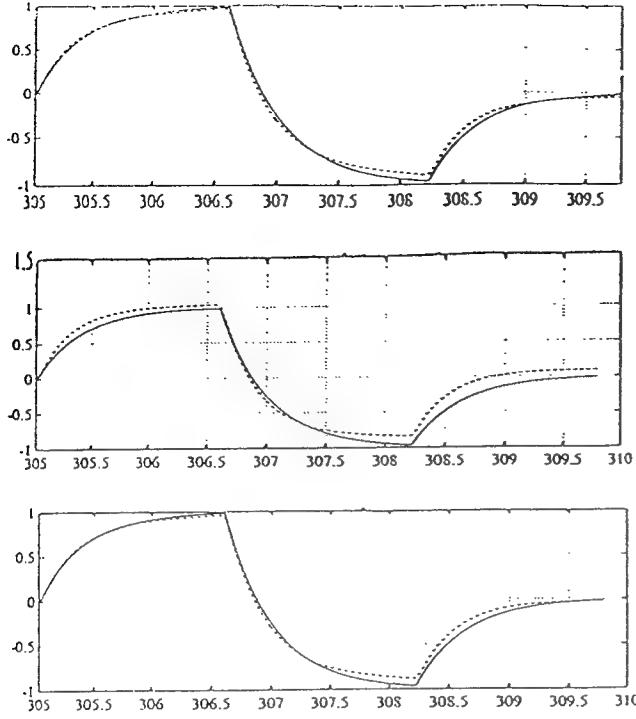


Figure 2: Unfailed Aircraft Step Responses. From top to bottom: pitch rate response to pitch command, roll rate response to roll command, and yaw rate response to yaw command.

the conditions are weaker. In the case of the indirect algorithm, the estimate of the high-frequency gain matrix must be nonsingular at all times for the control law to be implementable. For the direct input error algorithm, a proof of stability reveals the necessity for the adaptive parameter C_0 (as opposed to the estimate of CB in the indirect case) to remain nonsingular at all times. In [6], it was shown that a modification based on a sort of hysteresis could be incorporated in the algorithm with a least-squares update to guarantee stability of the overall adaptive system. The condition on the high-frequency gain matrix was only that an upper bound of its norm must be known.

• **Adaptation algorithms:** with the direct input error algorithm and with the indirect algorithm, one may use not only recursive gradient algorithms, but also least-squares algorithms which are faster and more efficient. Another advantage is that the least-squares algorithms can be used in their batch forms, which allow for monitoring of the quality of the estimation procedure [1].

• **Flexibility of control algorithm:** only the indirect approach allows for the use of alternate control strategies. One worthwhile option is the use of optimization algorithms which account for actuator saturation [5]. The direct algorithms rely on the model reference formulation and nontrivial modifications would have to be devised to account for actuator saturation. Another advantage of the indirect algorithm is also that it affords the opportunity to incorporate prior information about the plant parameters in the estimation procedure [4].

4 Implementation

4.1 Aircraft Model and Assumptions, and Design Considerations

Simulations were carried out using a detailed simulation of a twin-engine aircraft, developed at NASA-Dryden [2]. The model is a complete nonlinear aircraft simulation, including full envelope aerodynamics, atmospheric model,

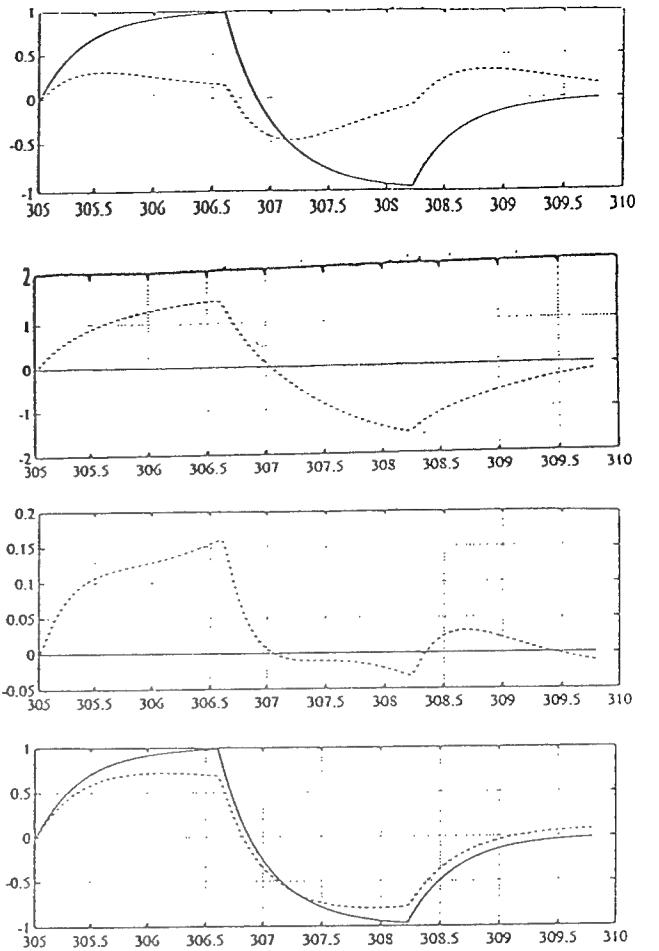


Figure 3: Failed Aircraft Step Responses with Original Controller. From top to bottom: pitch rate response to pitch command, roll rate response to pitch command, pitch rate response to roll command, roll rate response to roll command.

detailed engine dynamics, and actuator dynamics. The reconfigurable control system design, on the other hand, is based on the reduced-order model using the five states α, q, β, p, r . The control inputs are denoted δ_H, δ_A , and δ_R . There is a cross-feed between aileron command and elevator command. Specifically, the elevator command is the sum of the symmetric deflection δ_H and an antisymmetric deflection set to $\frac{1}{2}\delta_A$. For the reconfigurable control law, there is nothing that forces the same reduction of the five independent control surfaces to three control inputs (*i.e.*, it is not necessary to keep the same coupling matrix). However, it is convenient to keep the same structure for compatibility with the original control law, and for other considerations. While treating the five control surfaces as independent might lead to more flexibility in the reconfigurable control design, it would also require that these surfaces be actuated independently for the parameters of the B matrix to be identifiable.

The controlled outputs are chosen to be q, p, r . It was checked that this choice corresponded to minimum phase zeros for the flight condition under consideration. In general, it is possible to enforce minimum phase properties by replacing q and r by $q + K_\alpha \alpha$ and $r - K_\beta \beta$. A justification is that one has, approximately, $\dot{\alpha} = q$ and $\dot{\beta} = -r$ (although the precise location of the transmission zeros must be calculated with the coefficients of the A and B matr

a =

-0.7781	1.0130	0.0174	-0.0010	0.0191
-6.8269	-1.5534	2.0451	0.1328	0.6174
-0.9568	-0.2882	0.7456	0.1481	-0.7334
-6.1279	3.4852	-18.5242	-1.6079	3.9828
-0.7347	0.2993	5.2166	-0.0234	-0.1959

b =

-0.0111	0.0087	0.0110
-3.5152	2.3052	0.4689
-0.3959	-0.1977	-0.0802
-7.1580	13.0864	0.5626
-0.9876	0.7255	-2.3717

d =

3.5266
21.0414
3.2787
7.8556
0.4161

c0 =

-1.0660	0.1969	-0.1641
-0.5943	0.2983	-0.0467
0.2621	0.0093	-1.0001

g0 =

-2.4766	0.1488	2.6731	-0.0151	0.1008
-0.9055	-0.1853	2.7942	-0.0753	-0.2854
0.4445	0.0076	1.9412	-0.0266	0.8422

v =

8.3807
4.0727
-2.0685

Figure 4: Parameters for the Failed Aircraft

ces). The choice of q , p , and r as tracked outputs leads to a CB matrix that is a 9×9 matrix whose elements are the 3 moments created by each of the 3 control inputs. As long as the three vectors of moments are linearly independent (i.e., moments in all three directions can be independently created), the matrix CB is nonsingular, so that the assumptions under which the algorithms were derived are satisfied.

4.2 Experiments

The feasibility of the model reference control law was evaluated in experiments, using a diagonal reference model with elements $\frac{2.5}{s+2.5}$, following [7]. For the purposes of this paper, we simplified the problem by assuming that identification was performed before reconfiguration, using fairly good data segments. Work is in progress to study the complete problem, and the interaction between identification and control reconfiguration.

The aircraft parameters were identified using data segments of 10 seconds with rich excitation, and a batch least-squares algorithm. A standard flight condition at 9,800 ft and Mach 0.5 was chosen. The responses were obtained by having the aircraft under the control of a simple PID

regulator provided with the simulation package. Multiple steps of reference inputs were injected to provide excitation. The results are shown in Fig. 1. The angles are expressed in degrees and the angular rates in degrees per second. The elements of the matrices conform to expectations. For example, the A matrix has elements (1,2) close to 1 and (3,5) close to -1, with a large (4,3) element (effective dihedral). The B matrix is close to diagonal, which is to be expected for the unfailed aircraft. The values of the controller matrices were found using the relationships (7). The direct identification procedure based on the input error gave similar results, which are not shown.

Fig. 2 shows the step responses obtained with the model reference controller. A step was applied in the pitch axis, with a zero reference for the roll and yaw axes. Then, the experiment was repeated in a similar manner for the roll and the yaw axes. The first plot shows the pitch rate response to a step in pitch command. The dashed line is the aircraft response, while the solid line is the reference model response. In all three cases, the responses are found to be very satisfactory. The cross-axis responses were found to be small and are not reproduced, due to lack of space.

An interesting observation concerns the trim of the aircraft. During the identification, the aircraft was found to be trimmed by the PID controller at an angle of attack of about 4.6 degrees, requiring a command δ_H of approximately -2.86 degrees. The identification procedure did not identify the trim value for δ_H or α , but only the constant disturbance vector v . However, the trim value for δ_H can be calculated for the previous angle of attack to be $v(1) + G_0(1,1) \alpha_{trim}$ which is equal to $0.02 - 0.64 * 4.6 = -2.92$ and is remarkably close to the value obtained by the PID regulator. In other words, the least-squares procedure is successful in determining the trim value required to maintain level flight, without knowledge of what that flight condition actually is.

Fig. 3 shows the response of the aircraft after a failure where the left horizontal tail surface was stuck at its value for $t = 305$ seconds. The responses show significant degradation. Pitch response is much smaller than desired, and a significant roll rate response is incurred with the pitch command. These two effects are to be expected with the loss of about half the torque in the pitch axis and the loss of symmetry leading to a roll moment. Conversely, there is a small pitch rate response to a step in roll command and a reduced response in roll rate. These two effects can be explained by the cross-feed from roll command to differential elevator discussed earlier. The cross-feed leads to a pitching moment, and a partial loss of rolling moment is also incurred.

Fig. 4 gives the parameters identified for the failed aircraft. The same PID controller was used to obtain the data, although this controller was not able to maintain level flight for extended periods of time. The B matrix exhibits expected changes. The pitching effectiveness of the tail is reduced by a factor of 2, from -6.89 to -3.51. A large rolling moment of -7.15 also appears due to the asymmetric failure. The rolling moment effectiveness of the lateral channel is reduced from 18.72 to 13.08 due to the loss of effectiveness of the cross-feed to the elevator. A pitching moment of 2.30 also appears for the same reason. Note that the trim components d and v are significantly different from the values before the failure. This highlights the fact that, even if the aircraft remains stable after a failure, keeping the previous flight condition requires a new set of trim values that must be identified by the adaptive algorithm if the aircraft is to be prevented from rapidly diverging.

The first four plots of Fig. 5 show the responses of the

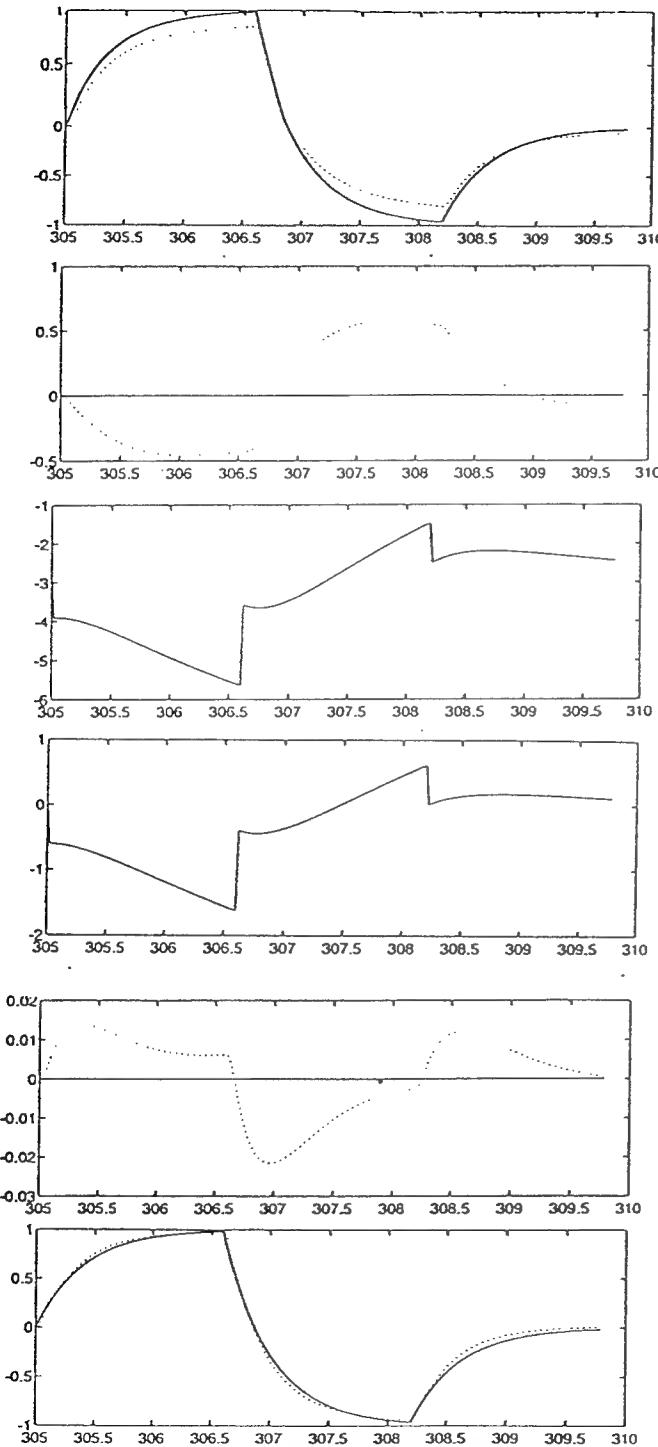


Figure 5: Failed Aircraft Step Responses with Reconfigured Controller. From top to bottom: pitch rate response to pitch command, roll rate response to pitch command, elevator response to pitch command, aileron response to pitch command, pitch rate response to roll command, roll rate response to roll command.

aircraft with the reconfigured control law to steps in the pitch command. The pitch rate response is found to be restored and the cross-response in the roll axis is found to be significantly reduced. The elevator and aileron commands are found to be remarkably similar. Trim value aside, one would expect $\delta_A = 0.58 \delta_H$ for the -7.15 and 13.08 terms in the B matrix to cancel each other. This is indeed what is observed. The last two plots of Fig. 5 show the responses to a roll command. Similarly to the results in the pitch axis, the cross-coupling in pitch is reduced and the roll response is improved. Here, one would expect $\delta_H = 0.66 \delta_A$, for the terms 2.30 and -3.51 in the B matrix to cancel. Plots for these variables are omitted for lack of space, but the relationship indeed holds.

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MULTIVARIABLE ADAPTIVE ALGORITHMS FOR RECONFIGURABLE FLIGHT CONTROL

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Abstract

The problem of reconfigurable flight control is investigated, focusing on model reference adaptive control and least-squares identification algorithms. The issue of parametrization is investigated, considering both direct and indirect algorithms. Different least-squares adaptation procedures are discussed. Results for the implementation of an input error direct adaptive control algorithm together with a detailed aircraft model are presented. An emphasis of the discussion is on the multivariable nature of the adaptive control problem. It is shown that strong cross-couplings appear after failures, and that the adaptive algorithm is able to restore decoupling between the rotational axes. Tracking of the commands and steady flight at the pre-failure condition are also achieved.

1 Introduction

Reconfigurable flight control has been a topic of research for several years. The goal is to create control systems that can compensate for failures and damages to aircraft. Such systems will increase the survivability of aircrafts, *i.e.* their ability to sustain damage while maintaining sufficient maneuverability. Reconfigurable flight control systems should also help to reduce costs. Indeed, redundant systems are currently used to provide back-ups for failed systems. Reconfigurable flight control systems may reduce the need for redundancy, and therefore lower aircraft costs.

Reconfigurable flight control systems compensate for failures using the unfailed portions of the control surfaces. If, for example, one of the airplane's elevators becomes jammed, the commands to the other elevator, as well as to the ailerons and to the rudder, are modified to compensate for that failure. The approach discussed in the thesis for the design of a reconfigurable control system is adaptive control. A simple view of an adaptive controller is the combination of a non-adaptive

control law, together with an identifier which uses the input to and the output from the plant to estimate unknown parameters. In indirect adaptive control, these parameters may be the matrices of a state-space representation, or the numerator and denominator polynomials of a transfer function representation. These estimates are used to select feedback gains to control the plant. In contrast, in direct adaptive control, the identifier estimates the feedback gains themselves. Since efficient algorithms are available from least-squares techniques to solve systems of linear equations, an important issue, for both direct and indirect approaches, is whether the problem can be cast in the form of a set of linear equations.

An adaptive algorithm has notable advantages over other methods. Failure identification methods are limited by what failures the system was designed to compensate for. If a failure occurs which is not one of those that had been anticipated, the system will not be able to identify the failure correctly, and the controller will not be reconfigured properly. Another advantage of adaptive methods is that they allow for reconfiguration not only for a broad range of failures, but also for varying flight conditions.

1.1 Reconfigurable Flight Control Systems

1.1.1 Failure Detection and Classification

In 1984, the U.S. Air Force began a program called the Self-Repairing Flight Control System (SR-FCS) Program [13]. The objectives of the program were to “improve the reliability, maintainability, survivability and life cycle cost” of aircraft [13, pg 504]. There were two main thrusts involved in achieving the goals of the program. The first method was the development of reconfigurable flight

control systems (FCS), meaning control systems that could be modified to compensate for changing conditions. The changing conditions consisted in a failure of the airplane's control surfaces or sensors, or in damage to the aircraft. Reconfigurable FCS would improve reliability and survivability. The second component of the program was the use of on-line diagnostics to identify failures. Improvements would come as reductions in maintenance and repair costs.

In developing the reconfigurable FCS, the Air Force decided on an approach based on identifying which component had failed, then modifying the FCS based on the identified problem. The failure was identified through a Failure Detection and Isolation (FDI) procedure. A local FDI algorithm was used to detect actuator failures, while a global FDI algorithm determined surface damage. The global FDI used a model of the aircraft to compare the measured output with the expected output. The error between the two was passed through several filters. The output of each filter represented a likelihood that the failure represented by the filter had occurred. These likelihoods were used by a Pseudo-Surface Resolver (PSR) to determine how control of the aircraft could be maintained. The PSR used a "modified pseudo-inverse that minimizes changes in control deflections after failure to maintain forces and moments." [13, pg 508] The maintenance diagnostics worked in the same manner.

There were several limitations to this approach. One of the more severe limitations was that any modeling error could be interpreted as a failure. Another limitation was that this method could only correctly identify errors that fell into a select few categories. However, the approach proved successful in the flight tests and was capable of handling various types of failures of the right stabilator. Urnes, Yeager & Stewart [29] concluded that "the test results of the Self-Repairing Flight Control System installed on an F-15 aircraft indicate high potential for the concepts evaluated."

Another major study of reconfigurable flight control systems was sponsored by NASA Langley and carried out by Alphatech (*cf.* [31]). The study considered the application of reconfiguration strategies to stable, commercial aircraft (Boeing 737). A single flight condition was assumed but a large variety of possible failures were simulated. The approach was the precursor of the SRFCS approach described above, consisting of two main components: a failure detection & identification module, and a control reconfiguration module. The control design in the NASA/Alphatech study was not based on the approximation of the unimpaired control actions, but on a redesign of the control law using linear quadratic (LQ) regulator theory. This approach is described in [18] and consists in specifying weighting matrices in an LQ problem so that the resulting closed-loop system satisfies some bandwidth constraints. Only changes in the B matrix of the state-space representation were considered, but the study included the possibility of incorporating knowledge of uncertainties in the estimated B matrix in the design.

The Alphatech project also studied extensively the problem created by changes in trim conditions, which act as constant disturbances to be added to the state-space model. An automatic trim algorithm was developed, based on an optimization procedure, and is reported in [32]. Of related interest is the work of Ostroff [24], which also considers the automatic control redesign for a Boeing 737 aircraft, but suggests the incorporation of integral action in the control law to solve the trim problem (essentially considering it as a disturbance rejection problem). Simulations for a mildly unstable aircraft model are also reported in this work.

Several other researchers have worked on design methods based on linear quadratic (LQ) techniques, assuming that the detection problem was solved independently. Huang & Stengel [16] presented an automatic redesign method based on implicit model following, incorporating integral action. Moerder *et al.* [21] studied the application of LQ controllers, but assumed that control gains were to be scheduled according to the decision of the failure detection and identification logic

(as opposed to being calculated in real-time).

Of related interest is the work of Maybeck & Stevens [19], which suggests a somewhat different approach. While assuming that the possible failures have been categorized, the method relies on a bank of Kalman filters to estimate the states of the system based on the different assumptions. Residual errors are used to calculate the probabilities of individual failures and the control input is the weighted average of the signals calculated under these respective assumptions. This is significantly different from the SRFCS approach where the control input corresponding to the most likely failure is chosen.

In [20], a multiple model adaptive estimation approach is used. Failures are represented as a vector of unknown system parameters, related to effectiveness of various actuators and sensors. Fault detection is achieved through the use of filters, where each filter is essentially a model of the VISTA F-16 in which a certain actuator or sensor has failed. How well the filter response and the airplane response match gives an estimate of how likely it is that the particular failure has occurred. If a partial failure has occurred, then the probability that the failure has occurred represents the loss of effectiveness of the sensor or control surface. Each failure has a set of control law gains associated with it. The controller combines the gains based on the probabilities generated. One of the difficulties of this approach is that there must be a non-zero input to the system. If not, false failures are sometimes detected, or real failures are not detected. Normally, the commands necessary to perform a maneuver are sufficient to excite the system. During steady level flight, however, there is not sufficient excitation. The solution to this problem was to add a small sinusoidal input to the system to ensure sufficient excitation.

1.1.2 Nonlinear and Adaptive Control

In contrast to the methods described above, several researchers have searched for methods that do not depend on the identification of the failures before taking action. A variety of directions have been pursued.

A logical first step consists in looking for a robust linear control law that would be satisfactory for all possible impaired aircrafts, and would achieve the required performance for the unimpaired aircraft. Schneider, Horowitz & Houpis [26] considered the use of quantitative feedback theory for that purpose. However, Chandler [6] illustrated with several examples that it is generally not possible to design a robust linear control law that guarantees stability for the impaired conditions while providing satisfactory performance for the nominal unfailed conditions. He advocated the design of a robust control law as a first line of defense to failures, giving time for the reconfigurable control law to take action, but implied that some form of reconfiguration, for example nonlinear or adaptive control, would be necessary.

Dittmar [11] investigated the use of an adaptive control approach based on hyperstability and the algorithms developed by Landau [17]. A simulation study concluded that the performance was equal or better than the SRFCS scheme, and could do so with less computer memory while accommodating a larger number of failure modes.

Morse & Ossman [22] also considered an adaptive control approach for the AFTI/F-16, using algorithms of Sobel & Kaufmann [27]. While they pointed out some problems with the specific theory that they were using, the authors developed their own design method for the selection of the parameters of the algorithm and showed that the method was successful even in the presence of multiple failures.

Gross & Migyanko [14] considered the use of "supercontroller" technology, which is a form

of nonlinear control based on polynomial networks. Coefficients of the polynomials were adjusted using an optimization program and a data base of optimal responses. This method was further developed in [1] and connected to recent work in neural networks. Sofge & White [28] also mention efforts currently under way at McDonnell Douglas in the neural network area. It is interesting to note that while the implementation of the controllers does not rely on explicit failure recognition, the training of the networks does, so that this method can be considered a hybrid between the two approaches discussed in this brief overview of the literature on reconfigurable flight control systems.

Research on the application of adaptive methods to reconfigurable control has also been carried out at Wright-Patterson Air Force Base [7], [8], [9], [10]. The emphasis of the research has been on the use of constrained least-squares identification methods and model predictive control. The results have been restricted to single-input single-output pitch axis models, but they have successfully included actuator rate saturation in the design, as well as prior information on the stability derivatives.

1.2 Overview

This thesis discusses some of the issues and problems with implementing multivariable adaptive control for reconfigurable flight control. Although we focus on model reference control, several formulations of the problem are discussed, as well as several adaptation algorithms. We discuss the linear parametrization of the aircraft dynamics, for both the direct and indirect approaches. The algorithms are implemented together with a complete nonlinear simulation of a twin-engine jet aircraft, and the performance of various approaches are evaluated and compared. The thesis expands on previous results presented in [4].

2 Adaptive Control Algorithms

2.1 Aircraft Model

The kinematic behavior of an airplane is governed by a set of nonlinear differential equations. For flight control system design, these equations can be approximated effectively by a set of linear differential equations.

$$\begin{aligned}\dot{x} &= Ax + Bu + d \\ y &= Cx\end{aligned}\tag{1}$$

A disturbance term d is included to account for the trim values of the input necessary to maintain flight at the operating point. By explicitly including this disturbance, the reconfigurable flight control system can automatically set the trim, freeing the pilot from having to. Because the trim values may change radically after a failure, such a feature is quite important.

The states of the aircraft are represented by x . The longitudinal states are α (angle of attack), q (pitch rate), h (altitude), and v (velocity). The lateral states are β (sideslip), p (roll rate), r (yaw rate), ϕ (roll angle), and ψ (yaw angle). For the design of stability augmentation flight control systems, the state vector can be reduced to only five states: α, q, β, p, r . The control inputs u are also divided into lateral and longitudinal inputs. The longitudinal inputs are δ_E (elevator command) and δ_T (thrust command). The lateral inputs are δ_A (aileron command) and δ_R (rudder command). In the aircraft under consideration in the simulations, the elevator command δ_E is interpreted as a symmetric command to the elevators, denoted δ_H , and a differential command to the ailerons, denoted δ_D . For stability augmentation flight control system design, δ_T is usually not considered.

There are several choices available for the control output y . The states q, p , and r are

good choices for low dynamic pressure and limited angle of attack (*cf.* [15]). The problem is then the control of a three-input, three-output, linear time-invariant system. Because of symmetry in the unfailed aircraft, the lateral and longitudinal axes are usually decoupled. However, after a failure the airplane will usually no longer be symmetric. Therefore, the lateral and longitudinal axes cannot be decoupled. The trim conditions may also change much faster at the time of a failure than they would in the unfailed aircraft, especially if a control surface becomes floating or missing.

2.2 Adaptive Control Algorithms

Several adaptive algorithms can be used in reconfigurable flight control. The control objective considered here is based on model reference control. A motivation is that this objective allows us to easily incorporate considerations of tracking and decoupling into design. Further, relatively simple algorithms are obtained. Several adaptation mechanisms have been proposed in the literature (see [25]). Those presented here are modified slightly to include the constant disturbance d , and to exploit the fact that all the states and their derivatives are measured. State variable filters are not needed to reconstruct the state and a direct state feedback control law can be used.

Assumptions and Reference Model

We consider the state-space model for the plant (1), where $x \in R^n$, $u \in R^m$, $y \in R^m$, and $d \in R^n$. We assume that the whole state x is available for measurement, although only the output y is to be tracked. The objective is for y to match the output y_M of a reference model

$$\dot{y}_M = A_M y_M + B_M r \quad (2)$$

where $y_M \in R^m$ and $r \in R^m$. The matrices A_M and B_M are arbitrary square matrices, with A_M stable. For the model reference control problem to have a relatively simple solution, we assume:

Assumption 1: The plant has relative degree 1, i.e. $\det(CB) \neq 0$.

Assumption 2: The plant transfer function is minimum phase, i.e. the zeros of transmission of the system are in the open left-half-plane.

The first assumption guarantees that the closed-loop transfer function of the plant can be made to match the transfer function of the reference model (2) using a proper compensator. If the assumption is not satisfied, the model reference control problem may still be solvable, but a more complex reference model would have to be chosen, so as to match the so-called *Hermite normal form* of the plant (cf. [25]). When the first assumption is satisfied, this Hermite form is simply $H(s) = \text{diag}\{1/s\}$ which means that the behavior of the plant at infinity is that of a multivariable integrator. The matrix CB is called the *high-frequency gain matrix* of the plant and is usually denoted K_P in the adaptive control literature. It is a most critical parameter for adaptive algorithms.

The second assumption is a necessary assumption to guarantee the internal stability of the model reference control algorithm. The dimension of the state-space model is m , while the dimension of the state-space model for the plant is n . Therefore $n - m$ modes must be made unobservable or uncontrollable. It can be shown that the model reference control law places m modes of the plant at the desired model reference locations, and makes the others unobservable by placing them at the locations of the transmission zeros.

Model Reference Control Law

We consider the state feedback control law

$$u = C_0 r + G_0 x + v \quad (3)$$

where $C_0 \in R^{m \times m}$, $G_0 \in R^{m \times n}$, $v \in R^m$ are free controller parameters. The control law is repre-

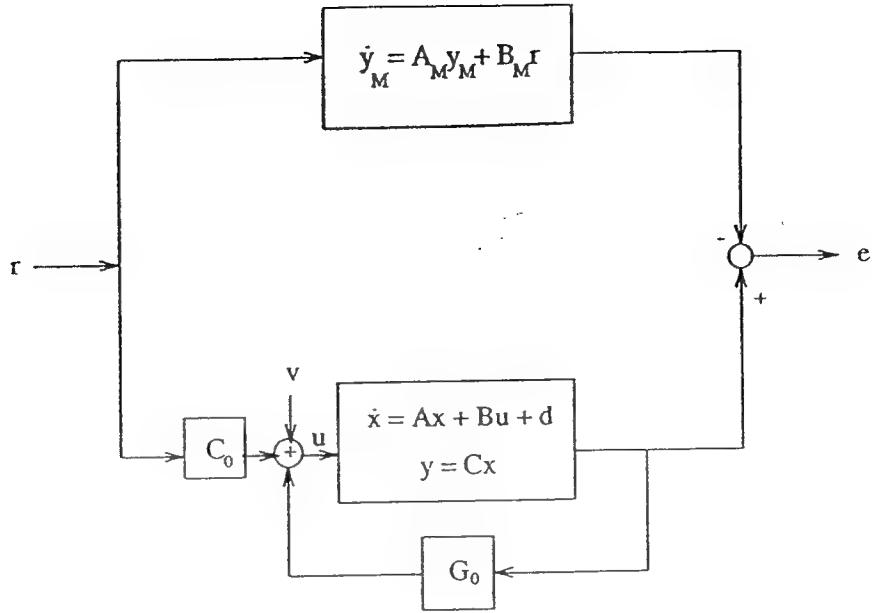


Figure 1: Model Reference Control Loop

sented in Figure 1. The closed-loop dynamics are given by

$$\begin{aligned}\dot{x} &= Ax + BC_0r + BG_0x + Bv + d \\ y &= Cx\end{aligned}\tag{4}$$

or, in terms of the output y ,

$$\dot{y} = (CA + CBG_0)x + CBC_0r + CBv + Cd\tag{5}$$

(5) leads to the same input/output relationship as that of the reference model (2) for the so-called *nominal* values of the controller parameters

$$\begin{aligned}C_0^* &= (CB)^{-1}B_M \\ G_0^* &= (CB)^{-1}(A_M C - CA) \\ v^* &= -(CB)^{-1}(Cd)\end{aligned}\tag{6}$$

If the plant was known, these would be the controller parameters that one would use to achieve the model reference control objective.

Indirect Adaptive Control Indirect adaptive control involves two stages. First, estimates of the plant parameters A , B , and d are generated. Once the plant parameters have been estimated, the estimates are used to generate controller parameters. If \hat{A} , \hat{B} , \hat{d} are the estimates of A , B , d , an error vector can be defined by

$$e_1 = \hat{A}x + \hat{B}u + \hat{d} - \dot{x} \quad (7)$$

Given the assumptions, the error vector can also be expressed as

$$e_1 = (\hat{A} - A)x + (\hat{B} - B)u + (\hat{d} - d) \quad (8)$$

A least-squares algorithm can be used to find the estimates of A , B , and d which minimize the sum of the norm of e_1 evaluated at several sampling instants. Each row of (7) can be treated independently in this process. The algorithm requires the measurement of \dot{x} . The derivative can often be obtained by filtered differentiation of x . For flight control systems, differentiation is not usually needed, as the derivatives can be measured directly.

Once the estimates are obtained, the controller parameters C_0 , G_0 , and v in (3) are obtained

from

$$\begin{aligned} C_0 &= (C\hat{B})^{-1}B_M \\ G_0 &= (C\hat{B})^{-1}(A_M C - C\hat{A}) \\ v &= -(C\hat{B})^{-1}(Cd) \end{aligned} \quad (9)$$

This is the same relationship as (6), replacing the plant parameters with their estimates. In addition to the issue of stability, a major question remains to be resolved: what to do when $C\hat{B}$ is singular.

There are few methods which satisfactorily address this problem, from a practical point-of-view, but it has been a subject of recent research in the adaptive control field.

Direct Adaptive Control Instead of estimating the plant parameters, a direct adaptive control algorithm estimates the controller parameters C_0 , G_0 , and v . There are two main approaches: output error and input error.

Output Error The output error $e_0 = y - y_M$ is defined as the difference between the plant output, and the output from the reference model, when given the same input as the plant. Because the system is assumed to have relative degree 1, there is no need for the so-called augmented error, and a simple algorithm results. The following fact is used.

Fact 1: *The output error e_0 satisfies*

$$e_0 = (sI - A_M)^{-1}(CB)[(C_0 - C_0^*)r + (G_0 - G_0^*)x + (v - v^*)] \quad (10)$$

Proof: Define

$$\dot{x}_M = A_M x_M + BC_0^* r + BG_0^* x_M + Bv^* + d \quad (11)$$

so that $y_M = Cx_M$ for some appropriate choice of initial conditions for x_M . Since $\dot{e}_0 = C\dot{x} - y_M = C\dot{x} - A_M C x_M - B_M r$, along with (6), we find

$$\dot{e}_0 = (CA + (CB)G_0^*)(x - x_M)(CB)[(C_0 - C_0^*)r + (G_0 - G_0^*)x + (v - v^*)] \quad (12)$$

which leads to (10).

Equation (10) can be expressed compactly in the form

$$e_0 = (sI - A_M)^{-1}(CB)[\Phi \cdot w] \quad (13)$$

where

$$\Phi = ((C_0 - C_0^*) (G_0 - G_0^*) (v - v^*)) \quad (14)$$

is the $m \times (m + n + 1)$ matrix of parameter errors, and the *regressor vector* $w^T = (r^T, x^T, 1)$ has dimension $m + n + 1$. The product $\Phi \cdot w$ is carried out in the time domain, and the resulting signal is applied to the linear time-invariant operator $(sI - A_M)^{-1}(CB)$.

Fact 2: The update law $\dot{\Phi} = -Ge_0w^T$ leads to an adaptive system that is stable in the sense of Lyapunov, with the property that e_0 tends to zero as $t \rightarrow \infty$, provided that $(sI - A_M)^{-1}$ is a strictly positive real transfer function matrix, and $(CB)^T G^{-1}$ is a positive definite matrix.

Proof: follows from Kalman-Yacubovich-Popov lemma (e.g. [23]).

This algorithm is similar to other algorithms, such as the one available in [23], with some adjustments needed because of the constant disturbance d and the state measurement. The strictly positive real condition can be satisfied by choosing A_M so that $A_M + A_M^T$ is negative definite. The other condition requires prior knowledge of the high-frequency gain matrix CB , and an appropriate choice for the adaptive gain matrix G .

Input Error Another formulation of the direct adaptive algorithm uses the following fact.

Fact 3: The following identity is satisfied for all time.

$$u = C_0^* B_M^{-1}(\dot{y} - A_M y) + G_0^* x + v^* \quad (15)$$

Proof: From (1), we find

$$\dot{y} - A_M y = C A x + C B u + C d - A_M C x \quad (16)$$

Using the nominal parameter values in (6), we get

$$\dot{y} - A_M y = (CB)(-G_0^* x + u - v^*) \quad (17)$$

from which (15) follows.

A new error equation can be defined from (15).

$$e_2 = C_0 B_M^{-1} (\dot{y} - A_M y) + G_0 x + v - u \quad (18)$$

Given the assumptions, (18) can be expressed as

$$e_2 = \Phi \cdot z \quad (19)$$

where Φ is the controller parameter error defined in (14), and z is a new regressor vector defined as

$$z = \begin{pmatrix} B_M^{-1} (\dot{y} - A_M y) \\ x \\ 1 \end{pmatrix} \quad (20)$$

The main difference between the error equations (19) and (13) is the absence of the transfer function between the parameter error and the error signal. This eliminates the strictly positive real condition necessary for the stability of the algorithm, including the condition on the high-frequency gain matrix. Also, it makes possible the use of least-squares algorithms.

The three algorithms presented above all achieve the same model reference control objective, but with different structures and different assumptions. The following issues should be considered:

Number of Parameters: Both direct methods estimate the controller parameters C_0 , G_0 , and v . These have a total of $m^2 + mn + m$ elements. The indirect algorithm estimates A , B , and d , which have a total of $n^2 + mn + n$ elements. Since $m < n$ in this application and in general, the direct algorithms estimate fewer parameters. For the reduced-order aircraft model, $n = 5$ and $m = 3$. The indirect algorithm must estimate 45 parameters, while the direct algorithms only estimate 27.

Prior Information: The direct output error algorithm requires the most restrictive assumptions, by imposing a positive definiteness condition on the product of the transpose of the high-frequency gain matrix with the inverse of the adaptation gain matrix. There is no obvious way to enforce this condition. For all practical purposes, this condition is a symmetry and positive definiteness condition on the high-frequency gain matrix itself, and is not easily guaranteed. The direct input error and indirect algorithms place less stringent conditions on the high-frequency gain matrix. The indirect algorithm requires that the estimate of the high-frequency gain matrix be nonsingular at all times. It can be shown that the direct input error algorithm requires that the parameter C_0 must be nonsingular at all times. In [12], it was shown that the stability of the overall adaptive system could be guaranteed by incorporating a modification based on a sort of hysteresis into the algorithm with a least-squares update, and requiring an upper bound on the norm of the high-frequency gain matrix.

Adaptation Algorithms: The indirect and direct input error approaches can be used with least-squares algorithms, in addition to gradient descent algorithms. The least-squares algorithms, which are faster and more efficient, can be used in their batch or recursive forms. The batch forms lend themselves to monitoring of the estimation quality [3].

Flexibility: The indirect algorithm is the most flexible of the three approaches. Other control strategies than model reference can be used, such as model predictive control [9]. The direct algorithms are not easily modified away from the reference model formulation. The indirect algorithm also has the flexibility to incorporate prior knowledge on the plant parameters.

Given these considerations, it was decided that the direct input error algorithm was best suited to the problem of reconfigurable flight control. Given the large number of computations required, it was felt that fewer parameters was a significant advantage. Also, the least-squares

algorithms are known to converge much faster than the gradient descent algorithms when the number of parameters is large. This is especially important considering that quick reconfiguration may be critical. A positive definiteness condition on the high-frequency gain matrix could also hardly be justified in this application. Therefore, the output error algorithm was not further considered. For the remainder of this thesis, it is implied that the direct input error formulation of adaptive control is being used.

Adaptation Algorithms

Given a set of linear equations with unknown coefficients, the least-squares algorithm constitutes a fast and efficient way to find the set of coefficients which most closely match the equations. The direct input error algorithm is easily formulated so that least-squares estimation can be used. Define the regressor vector z as in (20), and the parameter matrix θ as

$$\theta = \begin{pmatrix} C_0^T \\ G_0^T \\ v^T \end{pmatrix} \quad (21)$$

where C_0 , G_0 , and v are the control law parameters in (3). The input error e_2 can then be expressed as

$$e_2 = u - \theta^T z \quad (22)$$

where u is the system input.

Least-Squares Algorithm: We treat each component of e_2 independently. The problem then becomes that of finding a vector θ_i such that the sum of $|e_{2i}|^2$ over N sampling instants is minimized, where θ_i is the i^{th} column of θ , and e_{2i} is the i^{th} element of e_2 . The least-squares solution is

$$\theta_i = \left(\sum_{k=1}^N z[k]z^T[k] \right)^{-1} \left(\sum_{k=1}^N z[k]u_i[k] \right) \quad (23)$$

The solutions for the different columns of θ can be combined in a single expression.

$$\theta = \left(\sum_{k=1}^N z[k]z^T[k] \right)^{-1} \left(\sum_{k=1}^N z[k]u^T[k] \right) \quad (24)$$

The recursive least-square (RLS) algorithm can be derived from the batch solution. If we define

the *covariance matrix*

$$P[n] = \left(\sum_{k=1}^n z[k]z^T[k] \right)^{-1} \quad (25)$$

the batch least-squares for n sampling instants is

$$\theta[n] = P[n] \left(\sum_{k=1}^n z[k]u^T[k] \right) \quad (26)$$

From this, we derive

$$P[n+1] = P[n] - \frac{P[n]z[n+1]z^T[n+1]P[n]}{1 + z^T[n+1]P[n]z[n+1]} \quad (27)$$

$$\theta[n+1] = \theta[n] - \frac{P[n]z[n+1](z^T[n+1]\theta[n] - u^T[n+1])}{1 + z^T[n+1]P[n]z[n+1]} \quad (28)$$

Least-Squares with Forgetting Factor: The least-squares suffers from what is called the *covariance wind-up* problem [25]. Because the sum $\sum_{k=1}^n z[k]z^T[k]$ can grow without bound, the matrix P can become arbitrarily small. Adaptation will become very slow, which is unacceptable in the adaptive problem under consideration.

One solution to the covariance wind-up problem is to replace the least-squares algorithm with the least-squares with forgetting factor algorithm. The least-squares algorithm is modified by weighting the data exponentially, with older data having less weight than current data. The resulting algorithm is well known.

$$P[n] = \left(\sum_{k=1}^n z[k] z^T[k] \lambda^{n-k} \right)^{-1} \quad (29)$$

$$\theta[n] = P[n] \left(\sum_{k=1}^n z[k] u^T[k] \lambda^{n-k} \right) \quad (30)$$

$$P[n+1] = \frac{1}{\lambda} \left(P[n] - \frac{P[n] z[n+1] z^T[n+1] P[n]}{\lambda + z^T[n+1] P[n] z[n+1]} \right) \quad (31)$$

$$\theta[n+1] = \theta[n] + \frac{P[n] z[n+1] (u^T[n+1] - z^T[n+1] \theta[n])}{\lambda + z^T[n+1] P[n] z[n+1]} \quad (32)$$

Equations (29) and (30) represent the batch form of the least-squares with forgetting factor algorithm, while (31) and (32) are the recursive form. The parameter λ must be positive and less than 1. Typically, the range is restricted to 0.95 to 0.99. For 0.99, the equivalent time constant of the exponential weighting is equal to 100 samples.

Stabilized RLS with Forgetting Factor: The least-squares with forgetting factor algorithm unfortunately exhibits a new problem. The algorithm becomes unstable if there is insufficient excitation. This is often the case with aircraft, as steady level flight does not provide sufficient excitation for convergence of the parameters. The regressor vector z must be *persistently exciting* to guarantee convergence of the parameters to the nominal values [25]. One possible solution is to add a small perturbation to the controls, such as white noise. However, it will affect the flight of the aircraft (and may be a poor choice for a combat aircraft which is tracking a target, or while landing). Another solution is to use a stabilized version of the least-squares with forgetting factor. A new stabilized algorithm was recently proposed in [2] (based on a concept proposed in [30]) and is discussed now.

The stabilized least-squares algorithm is obtained by including an additional term in the error function used by the least-squares.

$$E(\theta[n]) = \sum_{k=1}^n (u^T[k] - z^T[k] \theta[n])^2 \lambda^{n-k} + \alpha |\theta[n] - \theta[n-1]|^2 \quad (33)$$

The additional term penalizes changes in the parameter matrix, θ . Setting $\frac{\delta E}{\delta \theta[n]} = 0$ yields

$$\theta[n] = \left(\sum_{k=1}^n z[k]z^T[k]\lambda^{n-k} + \alpha I \right)^{-1} \left(\sum_{k=1}^n z[k]u^T[k]\lambda^{n-k} + \alpha\theta[n-1] \right) \quad (34)$$

This is the equivalent of the batch solution of the least-squares, but it is not truly a batch solution, because of the dependence on $\theta[n-1]$. The matrix

$$P[n] = \left(\sum_{k=1}^n z[k]z^T[k]\lambda^{n-k} + \alpha I \right)^{-1} \quad (35)$$

is defined as the covariance matrix of this algorithm. A recursive form for the inverse of the covariance is

$$P^{-1}[n] = \lambda P^{-1}[n-1] + z[n]z^T[n] + \alpha(1-\lambda)I \quad (36)$$

with the initial condition $P^{-1}[0] = \alpha I$. The recursive formula for θ is

$$\theta[n] = \theta[n-1] + P[n]z[n](u^T[n] - z^T[n]\theta[n-1]) + \alpha\lambda P[n](\theta[n-1] - \theta[n-2]) \quad (37)$$

One problem is to transform (36) into a recursion for $P[n]$. The recursive least-squares and least-squares with forgetting factor algorithms make use of the matrix inversion lemma

$$(A + BC)^{-1} = A^{-1}B(I + CAB)^{-1}CA^{-1} \quad (38)$$

in the recursive update of $P[n]$. The inversion of the matrix $(I + CAB)$ is simplified because the product $z^T[n]P[n-1]z[n]$ is a scalar. For the update law (36), the matrix inversion lemma can still be used, but with $B = C^T$ defined as

$$B = \begin{pmatrix} z[n] & \sqrt{\alpha(1-\lambda)}I \end{pmatrix} \quad (39)$$

However the product $B^T P[n-1] B$ is not a scalar, but a matrix whose dimension is one greater than $P[n]$. It would be easier to update $P^{-1}[n]$ and invert it than to update $P[n]$.

An alternate solution is to replace $\alpha(1 - \lambda)I$ in the update law by $m\alpha(1 - \lambda)e_i e_i^T$, where e_i is a vector of zeros, except the i^{th} position which is one, and m is the dimension of z . As time progresses, i is incremented and returned to one when the end of the vector is reached. The matrix B becomes

$$B = \left(z \quad \sqrt{m\alpha(1 - \lambda)}e_i \right) \quad (40)$$

With this modification, the matrix $B^T P[n - 1]B$ is only 2×2 , which is easily inverted. Averaging analysis [2] shows that the averaged system responses are identical for both implementations.

A third option is a pseudo-batch recursive, or a periodic batch implementation. This algorithm updates the P^{-1} matrix each iteration, but updates θ only every N samples. In this way, P^{-1} is only inverted every N samples. The error criterion for this implementation is

$$E(\theta[n]) = \sum_{k=1}^{nN} (u^T[k] - w^T[k]\theta[n])^2 \lambda^{nN-k} + \alpha|\theta[n] - \theta[n-1]|^2 \quad (41)$$

where $\theta[n]$ is the parameter estimate based on nN data points. Solving $\frac{\delta E}{\delta \theta} = 0$, we find

$$\theta[n] = \left(\sum_{k=1}^{nN} z[k]z^T[k]\lambda^{nN-k} + \alpha I \right)^{-1} \left(\sum_{k=1}^{nN} z[k]u^T[k]\lambda^{nN-k} + \alpha\theta[n-1] \right) \quad (42)$$

Let $R[n] = P^{-1}[n]\theta[n]$. Then $P^{-1}[n]$ and $R[n]$ are updated according to

$$P^{-1}[n] = \lambda^N P^{-1}[n-1] + \alpha(1 - \lambda^N)I + \sum_{j=1}^N z[k]z^T[k]\lambda^{N-j} + \alpha I \quad (43)$$

$$R[n] = \lambda^N P^{-1}[n-1]\theta[n-1] + \alpha\theta[n-1] - \alpha\lambda^N\theta[n-2] + \sum_{j=1}^N z[n]u^T[n]\lambda^{N-j} \quad (44)$$

where $j = k - (n-1)N$. Every N samples, θ is updated by

$$\theta = P[n]R[n] \quad (45)$$

where $P[n]$ is computed by inverting $P^{-1}[n]$.

The stabilized recursive least-squares with forgetting factor has the property that the covariance matrix and its inverse are bounded [2]. The only condition is that $z[n]$ is bounded. The approximate algorithms using (40) was used in the thesis, in part because it only required the inversion of a matrix of size 2x2.

2.3 Reference Model and Auto Pilot

The objective of the model reference control law is for the airplane with feedback to have dynamics which approximate those of a chosen model. The reference model must have the same relative degree as the plant, but is otherwise arbitrary. The reference model chosen for the reconfiguration application is $H(s) = \frac{a}{s+a} I_3$, with $a = 2.5$ (cf. [15]). The input vector is $u^T = (\delta_H \ \delta_A \ \delta_R)$ and the output vector $y^T = (q \ p \ r)$.

If we assume that the reference model is matched, an auto pilot can also be designed around that model. One choice for an auto pilot is one that tracks the angles θ , ϕ , and ψ . Assuming that the reference model is matched, we have

$$\begin{aligned}\dot{q} &= -2.5q + 2.5q_c \\ \dot{p} &= -2.5p + 2.5p_c \\ \dot{r} &= -2.5r + 2.5r_c\end{aligned}\tag{46}$$

where q_c , p_c , and r_c are the elements of the reference input in (2). We can use the relationships

$$\begin{aligned}\dot{\theta} &= q \\ \dot{\phi} &= p \\ \dot{\psi} &= r\end{aligned}\tag{47}$$

with the commands

$$\begin{aligned}
 q_c &= g(\theta_c - \theta) \\
 p_c &= g(\phi_c - \phi) \\
 r_c &= g(\psi_c - \psi)
 \end{aligned} \tag{48}$$

so that the angles track the desired angles with the transfer function

$$\frac{2.5g}{s^2 + 2.5s + 2.5g} \tag{49}$$

With the constant g set to 1.6, the closed-loop poles are at $-1.25 \pm j 1.56$.

3 Implementation

3.1 AIAA Model

The algorithms discussed above were implemented on a FORTRAN simulation of a high performance, twin engine jet airplane [5]. The simulation includes nonlinear aerodynamics over the entire operational envelope, as well as thrust and engine response data. Some of the relevant parameters for the simulated aircraft are included in Table 1.

3.2 Batch LS Results

Our first implementation of model reference control was to use a batch least-squares identification off-line, with data collected from the AIAA simulation. Both the indirect and direct input error methods were used. Control performance was found to be similar in both cases, and the results shown here are for the direct input error algorithm. The identifications were performed at the flight condition of Mach 0.5 at altitude 9,800 feet. Identifications were also performed for an aircraft on which the left horizontal tail surface had become stuck. Ten seconds of data were used to perform

Parameter	Symbol	Value
Weight	W	45,000 lb
Wing Area	S	608.0 ft^2
Wing Span	b	42.8 ft
Mean Aerodynamic Chord	\bar{c}	15.95 ft
Moments		
	I_x	28,700 slug/ ft^2
	I_y	165,100 slug/ ft^2
	I_z	187,900 slug/ ft^2
	I_{xy}	0 slug/ ft^2
	I_{xz}	-520 slug/ ft^2
	I_{yz}	0 slug/ ft^2

Table 1: Parameters Used in the AIAA Simulation

each identification. The identified system matrices \hat{A} , \hat{B} , and \hat{d} were used to check the validity of the assumptions, in particular the invertibility of the high-frequency gain matrix and the minimum phase assumption. If the plant had not been minimum phase, a different choice of output could have been used to achieve minimum phase. Specifically, q and r could have been replaced by $q + K_\alpha \alpha$ and $r - K_\beta \beta$ [15].

The identified matrices C_0, G_0, v were then used to control the airplane. In turn, each of the reference inputs was given a series of step changes, while the other inputs were held at zero. The results for the direct input error identification are shown in Figure 2. The expected output y_M is represented by the solid lines, and the achieved output y is shown as dashed lines. The achieved output closely matches the expected output.

Figure 3 shows some of the responses from tests with the aircraft after failure, using the matrices identified for the unfailed aircraft. As one would expect, performance degrades, with a strong cross-coupling from the pitch axis to the roll axis. Figure 4 shows that the matrices identified

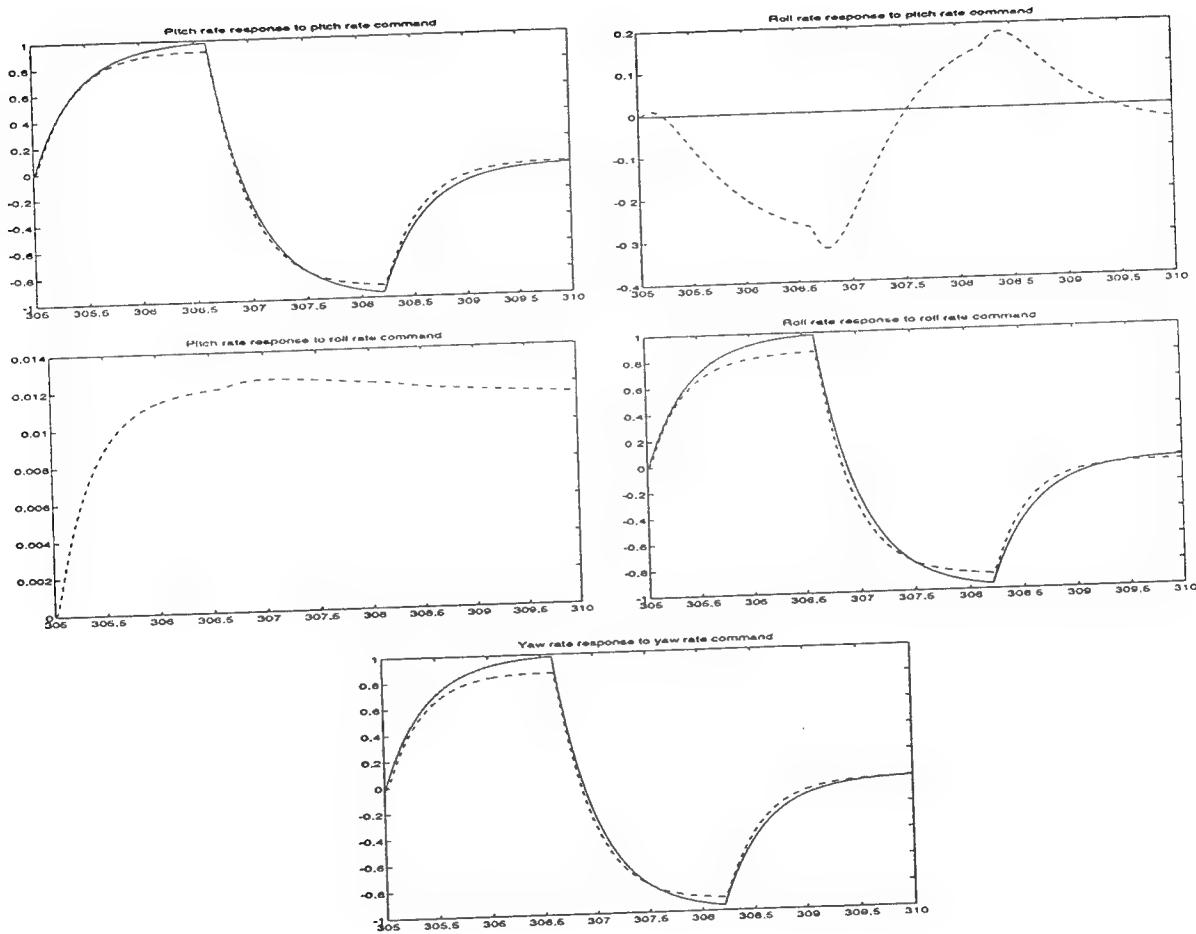


Figure 2: Batch LS: Responses for aircraft without failure

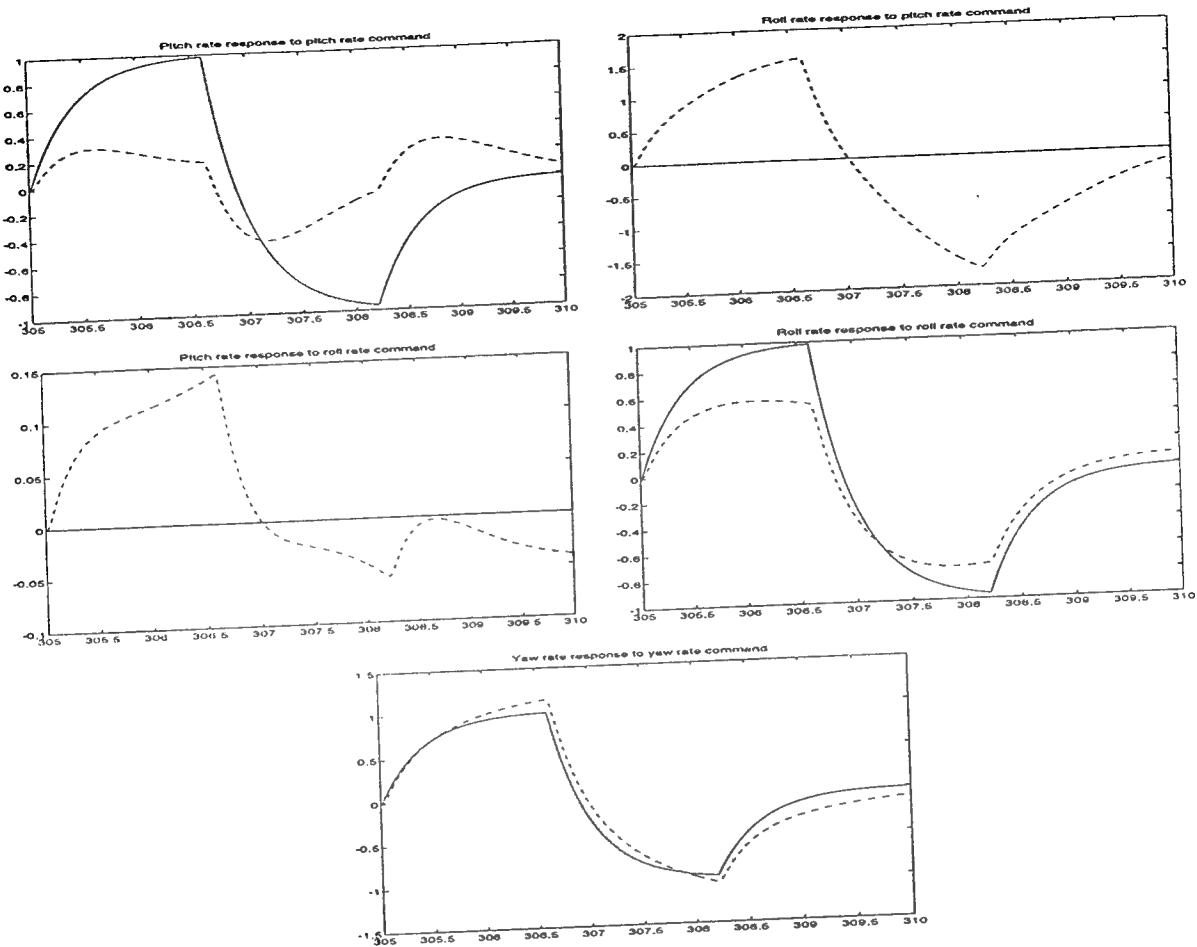


Figure 3: Batch LS: Responses for aircraft with failure, using parameters of unfailed aircraft

from the failed system compensate adequately for the failure. Performance degradation is minor, except for a residual cross-coupling from pitch to roll.

3.3 Recursive LS with Forgetting Factor Results

The next implementation was a recursive least-squares with forgetting factor (RLSFF) algorithm for direct input error adaptive control. Performance was tested in the same manner as it was tested for the off-line batch LS identification. In turn, each reference input was given a series of step

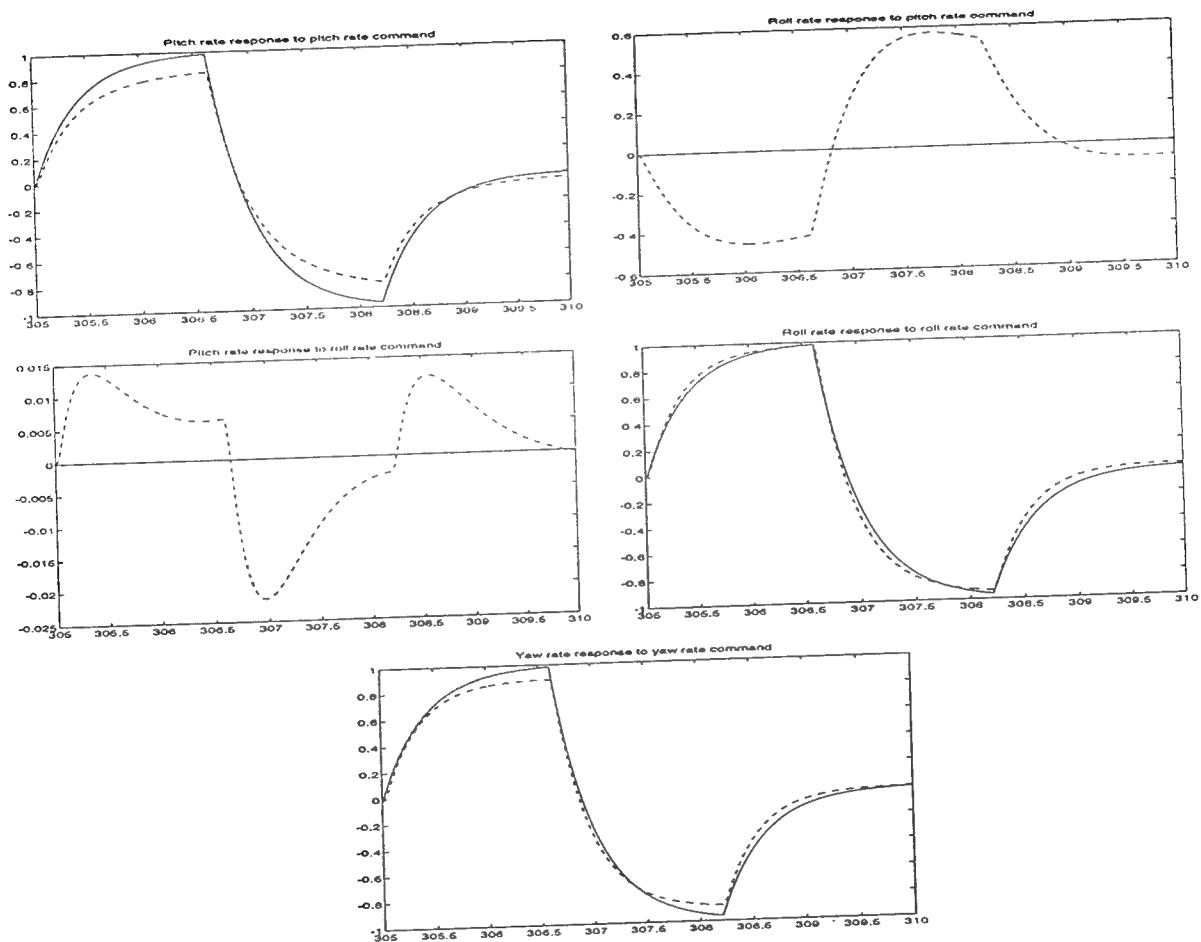


Figure 4: Batch LS: Responses for aircraft with failure

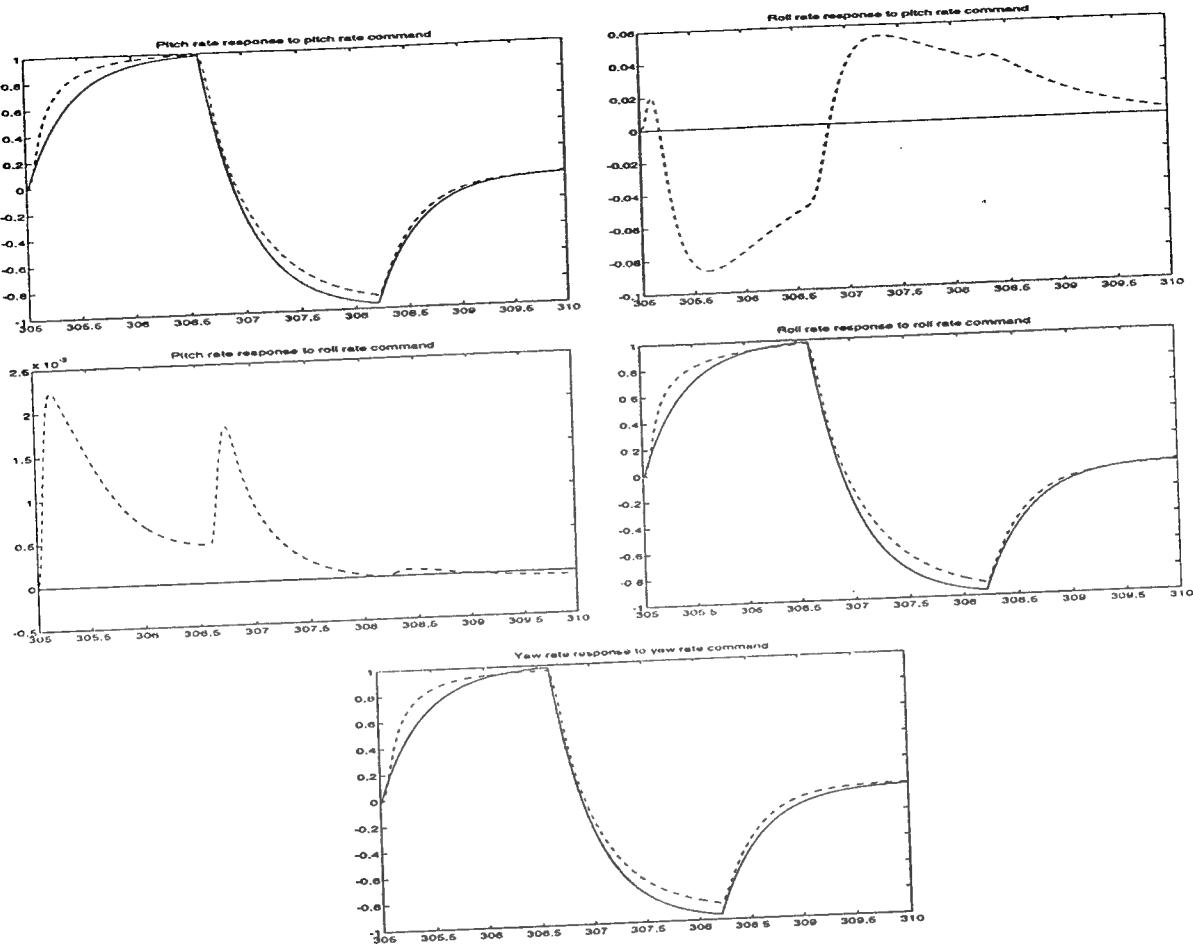


Figure 5: RLSFF: Responses for aircraft without failure

changes while the other inputs remained zero. The achieved output is compared to the output of the reference model. As can be seen in figure 5, the system output follows the reference model very well after a short transient. Figures 6 and 7 compare the results obtained with the batch LS algorithm (dashed lines) to those obtained from the RLSFF algorithm (dash-dot lines). The performance of the two algorithms is comparable, except that cross-couplings are significantly reduced with the recursive least-squares algorithm.

The main purpose of using a recursive algorithm is so that the control gains can adapt to

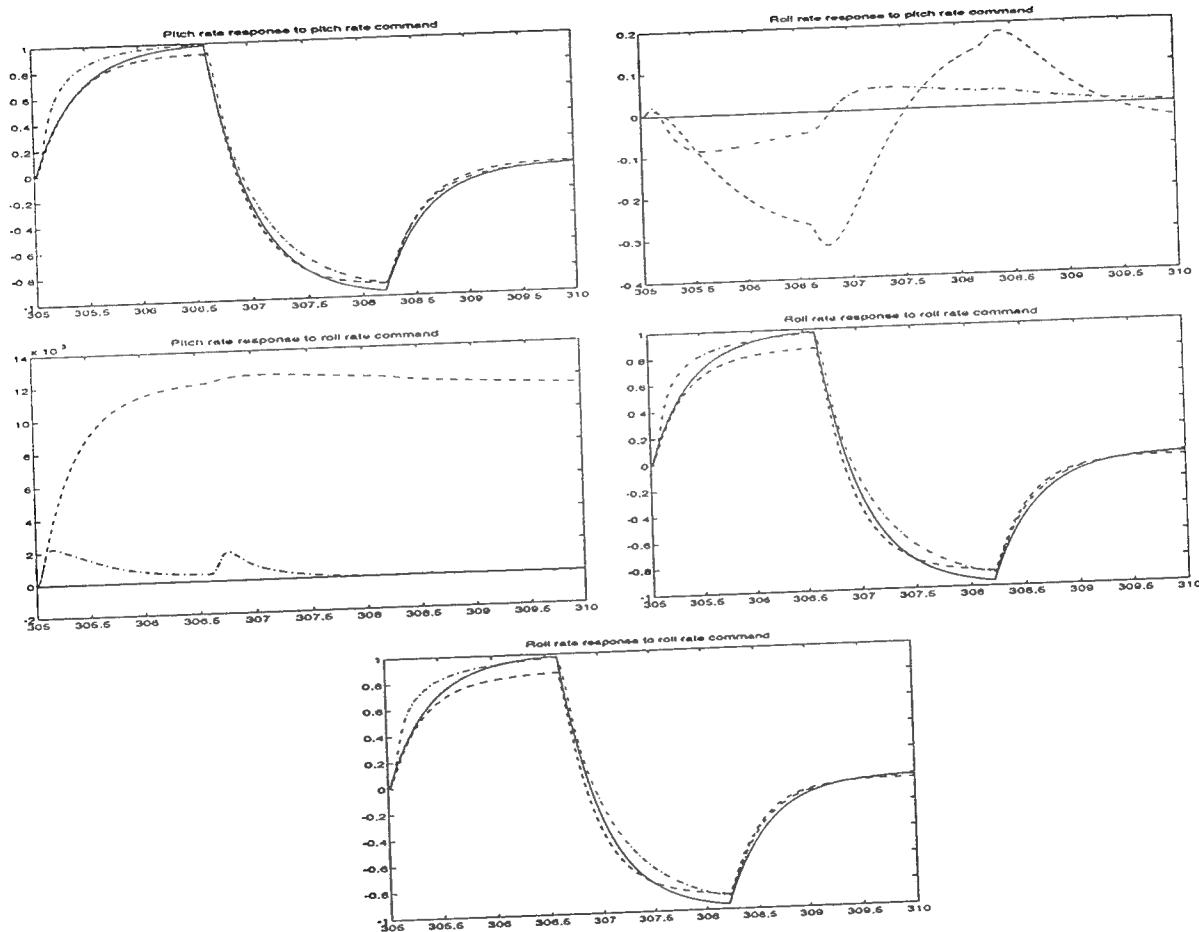


Figure 6: Comparison of batch and recursive LS results

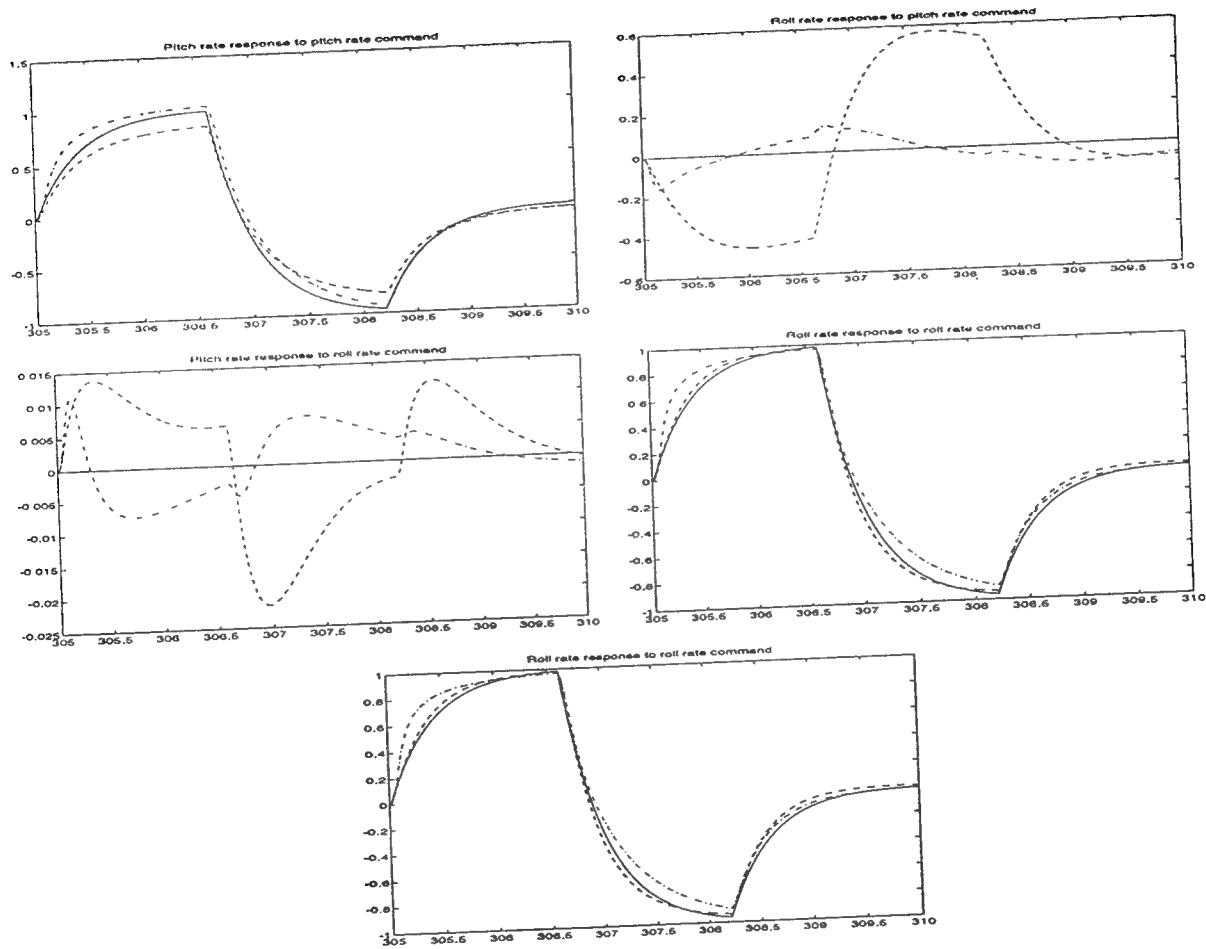


Figure 7: Comparison of batch and recursive LS results, aircraft with failure

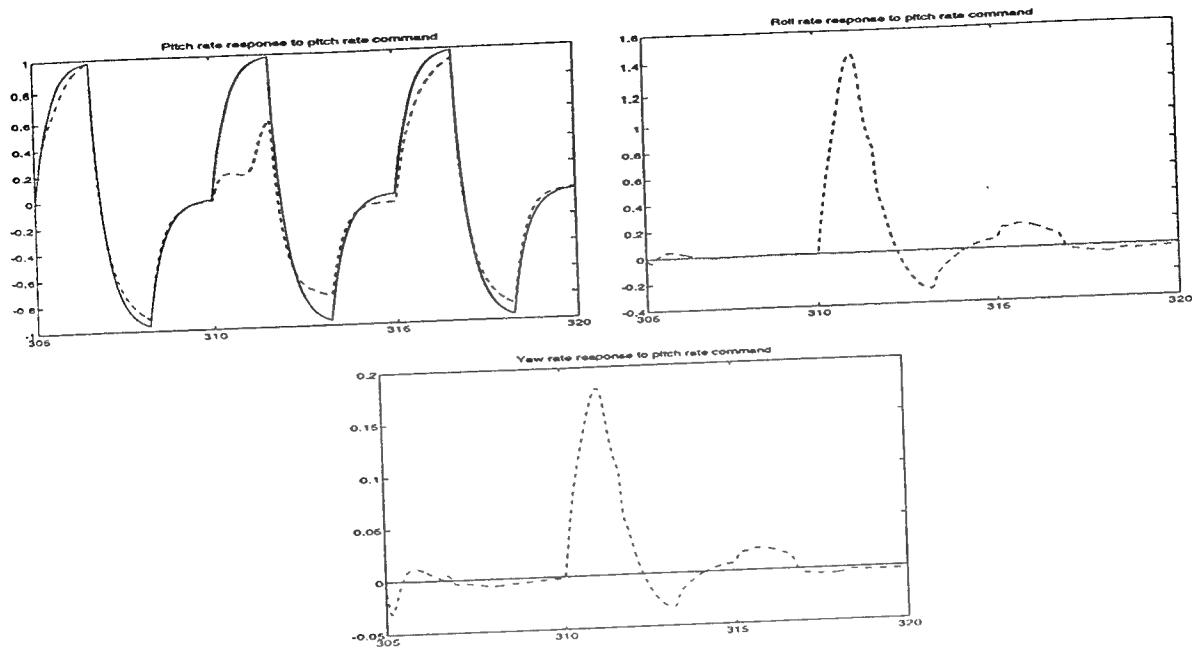


Figure 8: RLSFF, Adaptation to failure: failure occurs at $t = 310$ seconds.

changes in the plant. As before, a failure in the left horizontal tail surface was simulated. The RLSFF algorithm changes the control parameters to adapt to the failure, restoring system performance. Figure 8 shows pitch rate, roll rate, and yaw rate of the reference model (solid lines), and for the simulated airplane (dashed lines), when the pitch rate command was given step changes. The time period from 305 seconds to 310 seconds shows the response of the aircraft without failure, using the RLSFF identification of the control parameters. At 310 seconds, the left horizontal tail surface becomes stuck. Initially, much of the model tracking is lost, as seen from 310 seconds to 315 seconds. But, as seen from 315 seconds to 320 seconds, performance improves as the control parameters adapt to the changes in the plant.

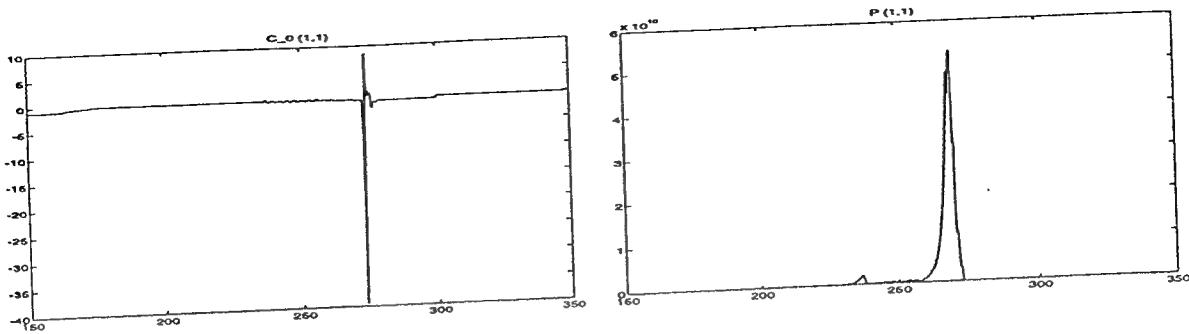


Figure 9: RLSFF: Low excitation results in instability

3.4 Stabilized RLSFF Results

The recursive least-squares with forgetting factor algorithm becomes unstable when there is insufficient excitation. Figure 9 shows the (1,1) elements of the parameter matrix C_0 and the covariance matrix P . During a period of insufficient excitation, such as during steady level flight, the matrices become unbounded. This instability is the motivation for the stabilized RLSFF algorithm.

In its unmodified form, the stabilized recursive least-squares with forgetting factor identification algorithm requires the inverse of a $m \times m$ matrix, where m is the number of elements of z . In our case $m = 9$. The modification proposed in equations (39) and (40) allows implementation of the algorithm using the inverse of a 2×2 matrix. This reduced algorithm was implemented on the airplane simulation. The test with the period of quiet was performed again, with the stabilized RLSFF. Figure 10 shows that the covariance and parameter matrices remain stable. Figure 11 shows the results of a similar test. In this experiment, a failure occurs in the middle of the period of quiet. During the period of quiet, the algorithm does not adapt to the failure, because there is no excitation which can be used to identify the changes in the parameters. But when there is excitation, the algorithm adapts to the failure.

One issue that presented itself was that numerical errors caused the covariance matrix update to become unstable. Enforcing the symmetry of P resolved this problem. Another solution

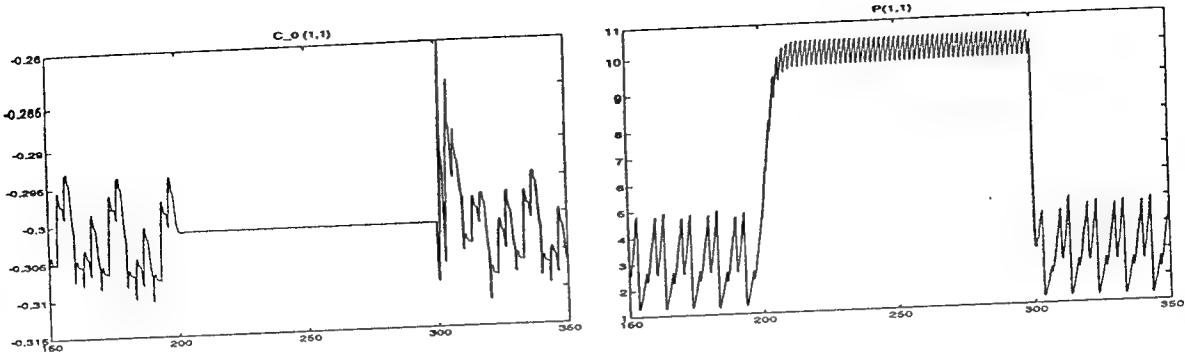


Figure 10: Stabilized RLSFF: Low excitation does not cause instability

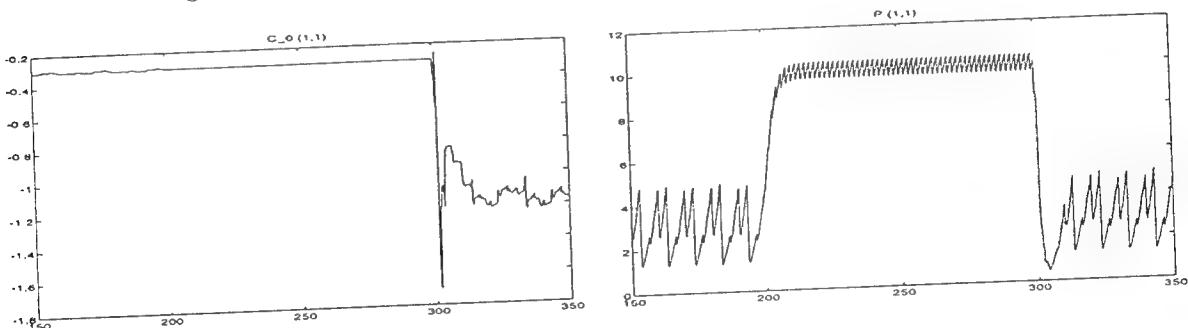


Figure 11: Stabilized RLSFF, aircraft with failure, at $t = 250$ seconds

would consist in using a square-root algorithm. This was not found to be necessary, however.

3.5 Auto Pilot Angle Tracking

An auto pilot was described in section 2.3 that could be used to control the angles θ , ϕ , and ψ of the airplane. This auto pilot was implemented with the batch LS identification matrices, the transfer function $\frac{4}{s^2+2.5s+4}$ describes the relationship from each angle command to the output angle.

The auto pilot was tested by holding two of the angle commands constant, while making step changes in the third command. Figure 12 shows some of the results from the tests. Note that

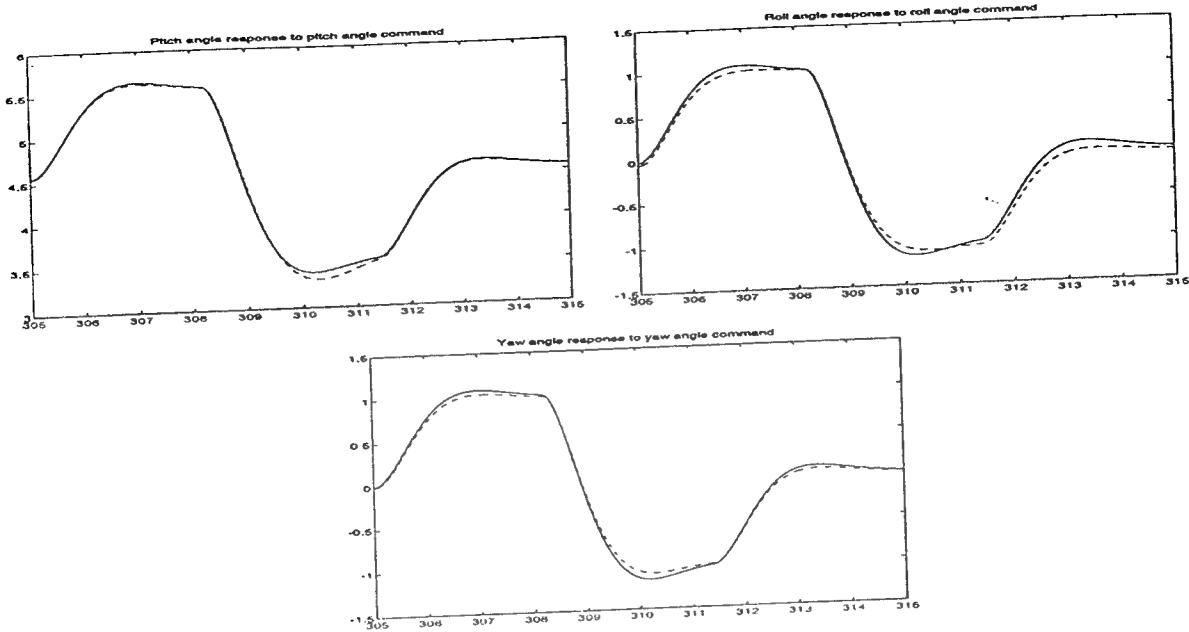


Figure 12: Tracking of θ , ϕ , and ψ by the auto pilot

there is a constant command for θ of 4.561 degrees. This is the angle of attack.

4 Conclusions

We had three main goals when designing a reconfigurable flight control system. First, we wanted a system that would select proper trim values for the inputs, even if a failure caused rapid changes in the nominal trim values. Second, we wanted the system to have its three inputs and three outputs decoupled. In the aircraft without a failure, this is easily achieved, because of symmetry. Once a failure occurs, the aircraft may lose its symmetry, making decoupling more difficult. Third, we needed a closed-loop system that would track the desired output. We have demonstrated that with model reference adaptive control it is feasible to achieve all three of these goals with satisfactory results.

The input error direct algorithm has important advantages over other algorithms. The

indirect algorithm requires more parameters to be identified, 45 versus 27. Since there are a large number of calculations in the identification algorithm, a large number of parameters is a handicap. The output error direct algorithm estimates the same number of parameters as the input error direct algorithm, but has more rigid stability conditions than either the indirect or input error direct algorithms. Also, least-squares adaptation algorithms can be used with the indirect and input error direct algorithms, but not with the output error direct algorithm. Least-squares algorithms have faster convergence for large numbers of parameters than gradient descent algorithms, which is critical for an reconfiguration.

The recursive least-squares with forgetting factor algorithm is often used in adaptive control. However, the algorithm suffers from instability if there is not sufficient excitation to the system. Since situations with little excitation are common, *e.g.* steady level flight, this is a serious issue. We have investigated a new algorithm which avoids this problem: the stabilized recursive least-squares with forgetting factor algorithm (SRLSFF). The SRLSFF achieves stability during periods of low excitation by penalizing in the error function changes in the parameter matrix θ . This change results in an algorithm with relatively weak conditions necessary for stability.

There are many related issues which remain to be studied. The performance of these algorithms should be tested when there is noise present in the system. Cursory testing has shown that the identification algorithms were not significantly affected by the noise, but control performance was degraded if the state variables were not filtered. Another issue is that there needs to be more testing, both with other failures, and at flight conditions throughout the operational envelope of the aircraft. Except as noted, all tests were conducted near a single operating point. At other flight conditions, it may be better to control the aircraft based on other criteria. One such criterion would be control of the aircraft based on the acceleration experienced by the pilot.

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Technical Notes and Correspondence

Multivariable Adaptive Control: Identifiable Parameterizations and Parameter Convergence

Michel de Mathelin and Marc Bodson

Abstract—The paper presents a direct model reference adaptive control algorithm for multivariable systems that has guaranteed parameter convergence properties. The algorithm requires fewer parameters than existing schemes, and the parameters converge to their nominal values under persistency of excitation conditions. Simple frequency domain conditions on the external inputs are derived to satisfy these conditions. The convergence of the controller parameters to their nominal values guarantees a certain degree of robustness of the algorithms and ensures that the model matching objective is satisfied asymptotically.

I. INTRODUCTION

The extension of single-input single-output adaptive control results to multivariable systems has been considered by several authors. To cite a few in the deterministic framework: Elliot and Wolovich [1], Singh and Narendra [2], and Dugard *et al.* [3]. The emphasis of these papers has been on finding parameterizations for the controllers, proving stability (in a few cases), and extending the schemes in ways to reduce the amount of *a priori* information required.

Close inspection reveals that parameter convergence is not guaranteed for these schemes, even with rich input signals, because they rely on parameterizations that do not define the parameters uniquely. The issue is the dual of the issue of *identifiability* in parameter estimation. A parameterization is a representation of a class of systems which associates a vector of parameters to each system. For linear time invariant systems, one usually says that a parameterization is identifiable if each transfer function corresponds to a unique parameter vector. Similarly, in direct adaptive control, the controller is parameterized so that a vector of controller parameters corresponds to each plant. Then, one may say that the parameterization is identifiable if the controller parameters are uniquely defined for each plant.

In this paper, we present a direct model reference adaptive control (MRAC) algorithm for multivariable systems, using a parameterization for which the parameters are uniquely defined. In addition, we give simple frequency-domain conditions on the external inputs to guarantee persistency of excitation and, hence, parameter convergence.

It is reasonable to ask why identifiable parameterizations would be useful in adaptive control. The first reason is that when nonidentifiable parameterizations are used in adaptive control, conditions of persistency of excitation on the regressor vectors cannot be satisfied

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in general, even with rich signals, so that the parameters may not converge to unique values. Since the set to which the parameters converge is usually unbounded, the convergence of the parameters can be very sensitive to disturbances. Small measurement noise and unmodeled dynamics can easily force convergence of the parameters to regions of the parameter space where the system becomes unstable. Conversely, with identifiable parameterizations and sufficiently rich inputs, a certain degree of robustness to noise and unmodeled dynamics is always guaranteed and the parameters will remain in the neighborhood of their nominal values (cf. [4]). An example presented in Section V illustrates the difference in robustness provided by an identifiable parameterization.

Another advantage of identifiable parameterizations is that they usually require a smaller number of parameters. This number already tends to be large for multi-input/multi-output systems and computational requirements grow fast with the number of parameters, especially for least-squares algorithms. On the other hand, a disadvantage is that the observability indices of the plant need to be known, as opposed to an upper bound.

An important contribution of this paper is the derivation of simple frequency domain conditions on the inputs that guarantee parameter convergence to the nominal values. Similar conditions were obtained, using generalized harmonic analysis, in multivariable recursive identification by de Mathelin and Bodson [5], and in single-input/single-output (SISO) adaptive control by Boyd and Sastry [6]. However, the conditions in multivariable adaptive control are not trivial extensions of the conditions in the SISO case. An important difference with the SISO case is that the necessary and the sufficient conditions for parameter convergence may be different. Indeed, parameter convergence may depend on the location of the spectral components of the inputs. For the same number of frequencies in the inputs, convergence may depend on the locations of these frequencies. This is not the case with SISO systems where only the number of frequencies is relevant.

II. MODEL REFERENCE CONTROL

Assume that the plant is of order n and is described by a square, nonsingular, strictly proper, and minimum phase transfer function matrix $P(s) \in \mathbb{R}^{p \times p}(s)$, the set of $p \times p$ matrices whose elements are rational functions of s . The following controller structure is considered

$$u = C_0 r + \Lambda^{-1} C[u] + \Lambda^{-1} D[y_p] \quad \text{with } r = M_0[r_0] \quad (1)$$

where u is the vector of inputs of the system, y_p the outputs, r the reference signals, $C_0 \in \mathbb{R}^{p \times p}$ is nonsingular, $M_0(s) \in \mathbb{R}^{p \times p}(s)$, and $\Lambda(s), C(s), D(s) \in \mathbb{R}^{p \times p}[s]$, the set of $p \times p$ matrices whose elements are polynomials in s . Further, $M_0(s)$ is a proper stable transfer function matrix, and $\Lambda(s)$ is a Hurwitz diagonal matrix, $\Lambda(s) = \text{diag}\{\lambda_i(s)\}$, such that $\Lambda^{-1}D$ is proper and $\Lambda^{-1}C$ is strictly proper.

By combining the equation of the plant, $y_p = P[u]$, with the equation of the controller (1), the output y_p can be expressed as

$$y_p = N_R((\Lambda - C)D_R - D_{NR})^{-1}\Lambda C_0[r] \quad (2)$$

where $\{N_R(s), D_R(s)\}$ is a right matrix fraction description (MFD) of $P(s)$, i.e., $P = N_R D_R^{-1}$.

Let the reference model $M(s) = H(s)M_0(s)$, where $H(s)$ is the Hermite normal form of $P(s)$ (see [4]), i.e., $\lim_{s \rightarrow \infty} H^{-1}(s)P(s) = K_p$ (high-frequency gain matrix) nonsingular and

$$H(s) = \begin{bmatrix} \frac{1}{(s+a)^{\nu_1}} & 0 & \cdots & 0 \\ \frac{1}{(s+a)^{\nu_2-1}} & \frac{1}{(s+a)^{\nu_2}} & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ & \vdots & \ddots & \frac{1}{(s+a)^{\nu_p}} \end{bmatrix}$$

with $\partial h_{ij}(s) < r_i - 1$

where a is arbitrary, but fixed *a priori*. The model output, y_m , is defined as

$$y_m = H M_0[r_0] = H[r]. \quad (3)$$

Then, it can be shown (cf. Appendix) that if $\partial \lambda_i \geq \nu - 1$ and $\nu \geq \nu_{\max}$ (where ν_{\max} is the maximum of the observability indices of $P(s)$), $\exists C_0^* \in \mathbb{R}^{p \times p}$, $C^*(s), D^*(s) \in \mathbb{R}^{p \times p}[s]$, solution of the Diophantine equation

$$N_R[(\Lambda - C^*)D_R - D^*N_R]^{-1}\Lambda C_0^* = H \quad (4)$$

such that model matching is achieved, $\Lambda^{-1}D^*$ is proper, and $\Lambda^{-1}C^*$ is strictly proper. In particular, $C_0^* = K_p^{-1}$ nonsingular, $\partial D^* \leq \nu_{\max} - 1$, and $\partial r_i C^* < \partial \lambda_i$ (where $\partial r_i C^*$ are the row degrees of C^*). An important result, whose proof is in the Appendix, is that the matrices C_0^*, C^*, D^* are unique if we add the following constraint on the solution

$$\partial c_i D^* \leq \nu_i - 1 \quad \forall i \quad (5)$$

where $\{\nu_i\}$ are the observability indices of $P(s)$ and $\partial c_i D^*$ are the column degrees of D^* .

III. ADAPTATION

The matching equality (4) is equivalent to

$$I = C_0^* H^{-1} P + \Lambda^{-1} C^* + \Lambda^{-1} D^* P. \quad (6)$$

Define $L(s) = \text{diag}\{l(s)\}$, with $l(s)$ Hurwitz, $\partial l(s) \geq d$, where d is the maximum degree of all elements of $H^{-1}(s)$. Then, multiplying both sides of (6) by L^{-1} and applying both transfer function matrices to u leads to

$$L^{-1}[u] = C_0^* (HL)^{-1}[y_p] + L^{-1}(\Lambda^{-1} C^*[u] + \Lambda^{-1} D^*[y_p]). \quad (7)$$

If $H(s)$ is known *a priori*, then (7) is an equation where the unknown parameters appear linearly. Indeed, define the matrices $C_1^*, \dots, C_{\nu-1}^*, D_1^*, \dots, D_{\nu}^* \in \mathbb{R}^{p \times p}$ such that

$$\Lambda^{-1} C^* = \sum_{i=1}^{\nu-1} C_i^* \frac{s^{(i-1)}}{\lambda(s)} \quad \Lambda^{-1} D^* = D_{\nu}^* + \sum_{i=1}^{\nu-1} D_i^* \frac{s^{(i-1)}}{\lambda(s)}.$$

The matrix of unknown controller parameters, $\theta^T = [C_0^* \dots C_{\nu-1}^* D_1^* \dots D_{\nu}^*]$ and the regressor vector

$$\psi^T = [(HL)^{-1}[y_p]]^T (\Lambda L)^{-1}[u]^T \dots s^{(\nu-2)} (\Lambda L)^{-1}[u]^T \\ (\Lambda L)^{-1}[y_p]^T \dots s^{(\nu-2)} (\Lambda L)^{-1}[y_p]^T L^{-1}[y_p]^T$$

so that the following error equation can be derived from (7)

$$e_2 = \theta^T \psi - L^{-1}[u] = (\theta^T - \theta^{*T})\psi = \phi^T \psi \quad (8)$$

where θ is the estimate of θ^* and ϕ is the parameter error. At this point, θ may be estimated using a standard linear estimation algorithm. The error equation is equivalent to the one presented in Elliot and Wolovich [1] and included in Sastry and Bodson [4]. It requires the *a priori* knowledge of the Hermite normal form $H(s)$ and an upper bound ν on the observability index ν_{\max} . Then, the number of parameters to identify, N_{θ} , is $2p^2\nu$. Note that, for stability considerations, other assumptions may be needed to guarantee that C_0^{-1} is bounded (cf. [7]).

To guarantee parameter convergence of the adaptive scheme, one finds that the uniqueness of θ^* must be guaranteed. From (5), the observability indices $\{\nu_i\}$ must be known and θ^* must be constrained so that $\partial c_i D^* \leq \nu_i - 1$ and $\nu = \nu_{\max}$. An advantage of uniqueness is that the number of parameters N_{θ} becomes smaller, $N_{\theta} = p^2\nu_{\max} + pn$. Note also that $N_{\theta} = 2pn$ if the observability indices are all equal and that $N_{\theta} = 2n$ in the SISO case, as expected.

IV. PARAMETER CONVERGENCE

Various approaches have been followed to guarantee stability of the adaptive scheme and, in particular, boundedness of all the signals. The main difficulty when trying to prove stability is that the estimate C_0 must stay bounded away from singularity. The simplest proofs assume that initial parameter estimates are available that are close to the nominal values. For larger initial errors, fixes are required to deal with the singularity regions of the algorithm. A parameter transformation to avoid singularity regions and a global stability proof are presented in de Mathelin [7]. For lack of space, we will not address stability issues here and proceed to study convergence properties assuming that stability is guaranteed.

The condition for parameter convergence is the well-known persistency of excitation (PE) condition. It states that if the regressor vector ψ is PE, then $\lim_{t \rightarrow \infty} \theta = \theta^*$ (exponentially, if the parameter estimation algorithm is a gradient algorithm or a recursive least-squares algorithm with covariance resetting or forgetting factor). The problem with such a condition is that it constrains time signals that are interrelated and not freely available to the designer. In particular, the condition may never be satisfied, as happens when the parameterization is not identifiable. Therefore, parameter convergence conditions are not of any real use unless they are translated into conditions on the external inputs of the adaptive system. This has been done, using generalized harmonic analysis, by Boyd and Sastry [6] in the SISO case and by de Mathelin and Bodson [5] in recursive multivariable identification. We use the same approach here to obtain the following results. Assume that the identifiable parameterization is used.

Theorem 1—Frequency Domain Conditions for Parameter Convergence-MRAC with K_p Known: Let the inputs r_i be stationary and uncorrelated (different frequencies in each input) and assume that the high-frequency gain matrix $K_p = C_0^{*-1}$ is known, so that the control parameter matrix C_0 does not need to be estimated.

- 1) If, for all i , the support of the spectral measure of the i th input, $S_{r_i}(\omega)$, contains at least $n + \nu_{\max} - p$ different values of ω , then ψ is PE.
- 2) If the support of the spectral measure of the inputs taken altogether, $S_R(\omega)$ does not contain at least $n + p(\nu_{\max} - 1)$ different values of ω , then ψ is not PE.

Theorem 2—Frequency Domain Conditions for Parameter Convergence-MRAC with K_p Unknown: Let the inputs r_i be stationary and uncorrelated (different frequencies in each input) and assume that the initial parameter error is sufficiently small that C_0 is nonsingular or that a parameter transformation is used to avoid singularity regions (see de Mathelin [7]).

TABLE I
SIMULATION RESULTS

Case	Inputs	f.c.	$\lambda_{\min}(R_{\psi_m}(0))$
1	$r_1 = \sin(2.0t) + \sin(4.0t)$ $r_2 = \sin(1.0t) + \sin(3.0t)$	4	$\neq 0$
2	$r_1 = \sin(2.0t) + \sin(1.19782411t)$ $r_2 = \sin(1.0t) + \sin(3.0t)$	4	0
3	$r_1 = \sin(1.0t) + \sin(3.0t)$ $r_2 = \sin(2.0t) + \sin(1.19782411t)$	4	0
4	$r_1 = 1.0 + \sin(2.0t)$ $r_2 = \sin(1.0t) + \sin(3.0t)$	3	0

- 1) If, for all i , the support of the spectral measure of the i th input, $S_{r_i}(\omega)$, contains at least $n + \nu_{\max} - p + 1$ different values of ω , then ψ is PE.
- 2) If the support of the spectral measure of the inputs taken altogether, $S_R(\omega)$ does not contain at least $n + p\nu_{\max}$ different values of ω , then ψ is not PE.

The proof of Theorem 2 is in the Appendix. The proof of Theorem 1 is a trivial simplification of the proof of Theorem 2. For the interpretation of the results, it should be noted that one sinusoid, $\sin(\omega_0 t)$, contributes to two frequency components, at $+\omega_0$ and $-\omega_0$. In the SISO case, the sufficient condition for parameter convergence is equal to the necessary condition. As expected (cf. [6]), $2n$ frequency components are necessary (with unknown k_p). Consequently, the number of frequency components necessary for parameter convergence is equal to the number of parameters N_θ . This is not true for multivariable systems where fewer frequencies are required than the number of parameters. Generally, in the multivariable adaptive control case, the sufficient condition is different from the necessary condition. A similar result was found in the identification case by de Mathelin and Bodson [5] where the sufficient condition was at least $n + \nu_{\max}$ frequency components per input and the necessary condition was $n + p\nu_{\max}$ frequency components in the inputs taken altogether. A particularly interesting fact is that when the necessary condition is satisfied but the sufficient condition is not, parameter convergence occurs in certain cases and not in others, depending on the location of the frequency components in the input. This peculiar phenomenon proper to multivariable systems is exposed through examples in the section to follow. Furthermore, it can also be shown (see [7]) that for some systems and for some particular input signals, parameter convergence occurs in the identification case but not in the MRAC case and vice versa. It is also interesting to note that there is no relationship between the particular frequencies and the zeros of the system. Indeed, there exists an infinite number of different input signals for which this particular phenomenon occurs, and it occurs in the MRAC case for different signals than in the identification case.

V. EXAMPLES

A. Example 1: MRAC Convergence

This example illustrates the results of Theorem 2. Let the plant be a two-input/two-output system of order four, with the following

transfer function matrix

$$P(s) = \begin{bmatrix} \frac{s^2+s}{s^3+3s^2+3s+2} & \frac{-2s^3-7s^2-7s-1}{s^4+4s^3+6s^2+5s+2} \\ \frac{1}{s^3+3s^2+3s+2} & \frac{s^3+3s^2+3s+1}{s^4+4s^3+6s^2+5s+2} \end{bmatrix}.$$

The system is stable and minimum phase. The observability indices are $\nu_1 = 2$ and $\nu_2 = 2$. Let

$$\Lambda(s) = \begin{bmatrix} (s+\lambda) & 0 \\ 0 & (s+\lambda) \end{bmatrix}$$

$$L(s) = \begin{bmatrix} \left(\frac{s}{l}+1\right) & 0 \\ 0 & \left(\frac{s}{l}+1\right) \end{bmatrix}$$

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ 0 & \frac{1}{(s+1)} \end{bmatrix}$$

with $\lambda > 0$ and $l > 0$. Therefore, the unknown parameter matrix is given by the equation at the bottom of the page, where the number of unknown parameter $N_\theta = 16$. The regressor vector is defined by

$$\psi^T = \begin{bmatrix} \frac{(s+1)y_{p_1}}{l(s)} & \frac{(s+1)y_{p_2}}{l(s)} & \frac{u_1}{l(s)(s+\lambda)} & \frac{u_2}{l(s)(s+\lambda)} \\ \frac{y_{p_1}}{l(s)(s+\lambda)} & \frac{y_{p_2}}{l(s)(s+\lambda)} & \frac{y_{p_1}}{l(s)} & \frac{y_{p_2}}{l(s)} \end{bmatrix}.$$

with $l(s) = \frac{s}{l} + 1$. Finally, the sufficient condition for parameter convergence is $n + \nu_{\max} - p + 1 = 5$ f.c. per input and the necessary condition is $n + p\nu_{\max} = 8$ f.c. in the inputs taken altogether.

To illustrate the results, it is convenient to compute the smallest eigenvalue of $R_{\psi_m}(0)$ (cf. proof of Theorem 2). This number is $\neq 0$ if ψ is PE and $=0$ otherwise. Table I summarizes the results obtained by computing $R_{\psi_m}(0)$ for several inputs. Cases 1, 2, and 3, with four frequency components (f.c.) in both inputs show that when the necessary condition is respected but the sufficient is not, parameter convergence will depend on the location of the f.c. in the input. In Case 1 there is parameter convergence. In Cases 2 and 3 the parameters do not converge to their true values. It was found that there exists an infinite number of values for the f.c. such that convergence does not occur, so that those values are not related in any obvious manner to the zeros of the system. Case 4 illustrates that eight f.c. in the whole input are absolutely necessary to have convergence.

We also simulated the behavior of the MRAC algorithm under these different inputs. To increase the speed of convergence, a least-squares algorithm with forgetting factor was used for the simulation. Fig. 1 shows the evolution of the function $v(t) = \ln\|\theta - \theta^*\|$ in Cases 1 and 2. When the input is not PE, all the estimates do not converge to their true value, and $\lim_{t \rightarrow \infty} v(t) = v(\infty) \neq -\infty$.

B. Example 2: Robustness

The following simulation illustrates the fact that the schemes using identifiable parameterizations generally have better robustness properties than those using nonidentifiable parameterizations. This example is one of several simulations that we ran, adding various types of noise and unmodeled dynamics to the ideal systems. In our experiments, we found several cases where both schemes (using identifiable and nonidentifiable parameterizations) were stable, and several cases (as the one presented here) where only the scheme based on an identifiable parameterization remained stable. The reverse was not observed.

$$\theta^{*T} = \begin{bmatrix} 1 & 2 & \lambda - 2 & 4 & \lambda^2 - 4\lambda + 3 & 2\lambda^2 - 5\lambda + 4 & 3 - \lambda & 3 - 2\lambda \\ 0 & 1 & -1 & \lambda + 1 & 1 - \lambda & \lambda^2 - 2\lambda + 2 & 1 & 1 - \lambda \end{bmatrix}$$

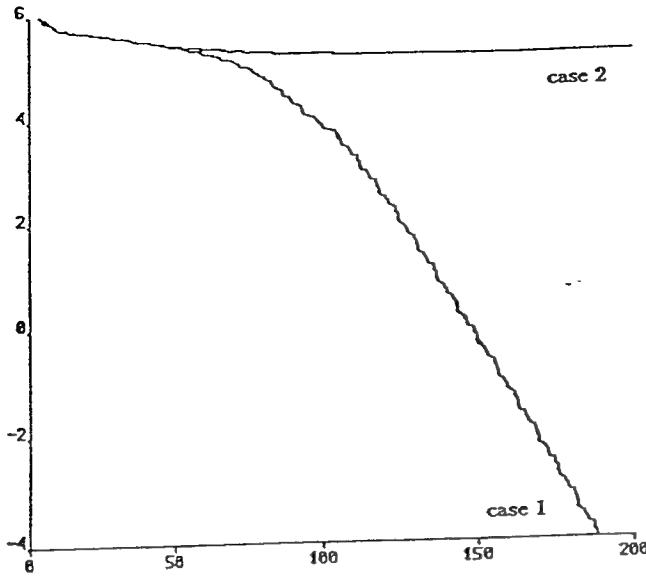


Fig. 1. $v(t)$ in case 1 and 2.

Identifiable Parameterization: $\partial c_i D^* \leq \nu_i - 1$: Assuming that K_p is known so that C_0 does not need to be identified, the unknown parameter matrix is given by

$$\theta^{*T} = \begin{bmatrix} 2\lambda & 2+\lambda & \lambda^2 - 2\lambda + 7 & 4-\lambda & -1-\lambda \\ 1-2\lambda & -1-\lambda & -\lambda^2 + 3\lambda - 10 & -6+2\lambda & 2+\lambda \end{bmatrix}$$

where the number of unknown parameters $N_\theta = 10$. The regressor vector is defined by

$$\psi^T = \begin{bmatrix} \frac{u_1}{l(s)(s+\lambda)} & \frac{u_2}{l(s)(s+\lambda)} & \frac{y_{p_1}}{l(s)(s+\lambda)} \\ & & \frac{y_{p_2}}{l(s)(s+\lambda)} & \frac{y_{p_1}}{l(s)} \end{bmatrix}.$$

From Theorem 1, the necessary condition for parameter convergence is three f.c. per input and the necessary condition is five f.c. in the inputs taken altogether.

Nonidentifiable Parameterization: $\partial c_i D^* \leq \nu - 1 = \nu_{\max} - 1$. The unknown parameter matrix is given by

$$\theta_n^{*T} = \begin{bmatrix} \theta^{*T} & 0 \\ 0 & \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} \end{bmatrix} \begin{bmatrix} 1 & 1 & \lambda - 3 & \lambda - 2 & -1 & -1 \end{bmatrix}$$

where k_1 and k_2 are arbitrary constants. Therefore, $N_\theta = 12$ and the regressor vector is defined by

$$\psi_n^T = \begin{bmatrix} \psi^T & \frac{y_{p_2}}{l(s)} \end{bmatrix}.$$

A simulation of the MRAC algorithm was made for both parameterizations with the following PE reference input: $r_1 = \sin(2 * t) + \sin(4 * t)$ and $r_2 = \sin(t) + \sin(3 * t)$ and with unmodeled dynamics and high-frequency output noise added to the nominal system. The unmodeled dynamics were $(144/(s+12)^2)$ on the first output y_{p_1} , and $(289/(s+17)^2)$ on the second output y_{p_2} . The noise was an additive output noise equal to $0.1 \sin(20 * t)$ in the first output and equal to $0.1 \sin(25 * t)$ in the second output. The unmodeled dynamics and the noise are more than a decade away from the dynamics of the nominal system and of the reference model. Furthermore, the average amplitude of the noise is about 5% of the average amplitude of the reference output. For the simulations, a least-squares algorithm with forgetting factor was used to increase the speed of convergence. Also, a stabilizing term was added to the least-squares algorithm, see Kreisselmeier [9], to prevent divergence of the covariance matrix of the least-squares algorithm (which can occur with the nonidentifiable parameterization). The parameters, λ and l , were selected equal to four and 10, and the initial controller parameters were chosen equal to zero. The norm of the output error, $\|e_0(t)\|$, is shown in Fig. 2 for the identifiable parameterization and in Fig. 3 for the nonidentifiable parameterization. Clearly, the system becomes unstable with the nonidentifiable parameterization and remains stable with the identifiable one (much longer simulations were run to verify this fact).

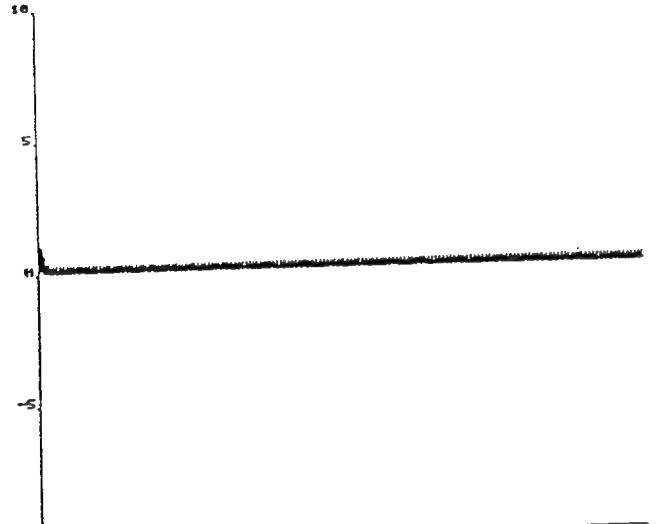


Fig. 2. $\|e_0(t)\|$ with identifiable parameterization.

Let the nominal plant be a two-input/two output system of order 3, with the following transfer function matrix

$$P(s) = \begin{bmatrix} \frac{2s^2+3s-4}{s^3+3s^2+4s+2} & \frac{s^2-s-8}{s^3+3s^2+4s+2} \\ \frac{-s^2-4s+7}{s^3+3s^2+4s+2} & \frac{s+13}{s^3+3s^2+4s+2} \end{bmatrix}.$$

The system is stable and minimum phase. The observability indices are $\nu_1 = 2$ and $\nu_2 = 1$. Let

$$\Lambda(s) = \begin{bmatrix} (s+\lambda) & 0 \\ 0 & (s+\lambda) \end{bmatrix}$$

$$L(s) = \begin{bmatrix} \left(\frac{s}{l} + 1\right) & 0 \\ 0 & \left(\frac{s}{l} + 1\right) \end{bmatrix}$$

$$M(s) = \begin{bmatrix} \frac{1}{(s+1)} & 0 \\ 0 & \frac{1}{(s+1)} \end{bmatrix}$$

with $\lambda > 0$ and $l > 0$.

APPENDIX

Lemma 1—Null Matrix Condition: Given $\{N_R, D_R\}$, a right MFD of a $(p \times m)$ strictly proper system with observability indices $\{\nu_i\}$. If $D(s) \in R^{q \times p}[s]$, $N(s) \in R^{q \times m}[s]$, with q arbitrary, are such that

$$D(s)N_R(s) = N(s)D_R(s) \quad \forall s \text{ and } \partial c_i D \leq \nu_i - 1 \quad \forall i$$

then $D(s) = 0$ $N(s) = 0$.

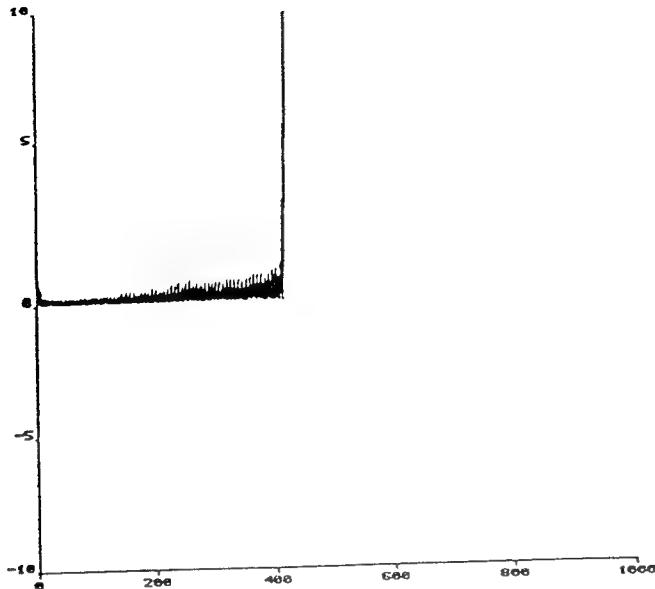


Fig. 3. $\|e_0(t)\|$ with nonidentifiable parameterization.

Proof: Define $D_i(s)$ and $N_i(s)$ as the i th row of $D(s)$ and $N(s)$, then by hypothesis

$$D_i(s)N_R(s) = N_i(s)D_R(s) \quad \text{and}$$

$$\partial c_j D_i = \partial D_{ij} \leq \nu_j - 1 \quad \forall j \quad \forall i.$$

Suppose that $\{N_L, D_L\}$ is the canonical left MFD (cf. [9]), i.e.,

$$\partial c_j D_L = \partial r_j D_L = \nu_j \quad \Gamma_c[D_L] = I$$

$$D_L(s)N_R(s) = N_L(s)D_R(s)$$

where $\Gamma_c[D_L]$ is the *leading column coefficients matrix* of D_L . Now, if $\nu_k = \nu_{\max}$, then the matrices $\{N_L^*, D_L^*\}$ defined by

$$\begin{aligned} \text{jth row of } D_L^* &= \text{jth row of } D_L \quad \forall j \neq k \\ \text{jth row of } N_L^* &= \text{jth row of } N_L \quad \forall j \neq k \\ \text{kth row of } D_L^* &= \text{kth row of } D_L + D_i \\ \text{kth row of } N_L^* &= \text{kth row of } N_L + N_i \end{aligned}$$

form another left MFD ($\forall i$) with

$$\partial c_j D_L^* = \partial r_j D_L^* = \nu_j \quad \Gamma_c[D_L^*] = I$$

$$D_L^*(s)N_R(s) = N_L^*(s)D_R(s).$$

Given the uniqueness of the canonical left MFD

$$\begin{aligned} D_L^*(s) = D_L(s) &\Rightarrow D_i(s) = 0 \quad N_i(s) = 0 \quad \forall i \\ &\Rightarrow D(s) = 0 \quad N(s) = 0. \end{aligned} \quad \square$$

Model Reference Matching Equality: We first show that there exists a solution to the matching equality (4)

$$\begin{aligned} N_R[(\Lambda - C^*)D_R - D^*N_R]^{-1}\Lambda C_0^* &= H \\ \Leftrightarrow C_0^*H^{-1}P &= I - \Lambda^{-1}C^* - \Lambda^{-1}D^*P. \end{aligned}$$

Let $\{N_L, D_L\}$ be a left coprime MFD of $P(s)$ with D_L row reduced, then the *leading row coefficients matrix* of D_L , $\Gamma_r[D_L]$, is

nonsingular and $\partial D_L = \nu_{\max}$ (cf. Kailath [10]). Divide $\Lambda K_p^{-1}H^{-1}$ on the right by D_L . Then, by the polynomial matrix division theorem (see [10, pp. 388–390]), $\exists Q, R \in \mathfrak{R}^{p \times p}[s]$ such that $\Lambda K_p^{-1}H^{-1} = QD_L + R$ and RD_L^{-1} is strictly proper. Furthermore, $\partial c_i R < \partial c_i D_L \leq \nu_{\max}$. Then, let

$$D^* = -R = QD_L - \Lambda K_p^{-1}H^{-1} \quad C^* = \Lambda - QN_L$$

$$C_0^* = K_p^{-1}.$$

It is easy to see that the given C_0^*, C^*, D^* solve the matching equality. Furthermore, since $\partial \lambda_i = \nu - 1$, $\Lambda^{-1}D^*$ is proper. On the other hand

$$\begin{aligned} \lim_{s \rightarrow \infty} \Lambda^{-1}C^* &= \lim_{s \rightarrow \infty} (I - K_p^{-1}H^{-1}P - \Lambda^{-1}D^*P) \\ &= I - I = 0 \end{aligned}$$

so that $\Lambda^{-1}C^*$ is strictly proper and $\partial r_i C^* \leq \partial \lambda_i - 1$, e.g., $\partial r_i C^* \leq \nu_{\max} - 2$ if $\partial \lambda_i = \nu_{\max} - 1$. To insure the uniqueness of the solution to the Diophantine equation, we choose for D_L the canonical left MFD, so that $\partial c_i D_L = \nu_i$ and $\partial c_i D^*$ becomes $\leq \nu_i - 1$. Then, suppose that $C_0^* + \Delta C_0, D^* + \Delta D, C^* + \Delta C$ is another solution to the matching equality. Then

$$\Delta C_0 H^{-1}P + \Lambda^{-1}\Delta C + \Lambda^{-1}\Delta D P = 0$$

$$\lim_{s \rightarrow \infty} \Delta C_0 H^{-1}P = \Delta C_0 K_p = 0 \Rightarrow \Delta C_0 = 0$$

$$\Rightarrow \Delta C D_R + \Delta D N_R = 0 \quad \text{and} \quad \partial c_i \Delta D \leq \nu_i - 1$$

by the null matrix condition (Lemma 1), $\Delta C = \Delta D = 0$ and the solution is unique. \square

Proof of Theorem 2: Define the transfer function matrix, $H_{\psi u}(s)$, as the transfer function between ψ and u , and define the model signals ψ_m as the signals ψ when the parameter error $\phi = 0$

$$\psi_m = H_{\psi u} P^{-1} H[r] = H_{\psi_m r}[r]. \quad (9)$$

The proof follows steps similar to those of the proof of Theorem 1 in de Mathelin and Bodson [5] and can be found in de Mathelin [7]. The outline of the proof is as follows. First, it is shown that ψ PE is equivalent to ψ_m PE. Then, since the reference input r is assumed stationary ψ_m PE is equivalent to $R_{\psi_m}(0) > 0$, where $R_{\psi_m}(t)$ is the autocorrelation of $\psi_m(t)$. Finally, the first part of the theorem is proved by showing that $R_{\psi_m}(0) > 0$ when the sufficient condition is verified and the second part by showing that $R_{\psi_m}(0)$ is singular when the necessary condition is not verified. \square

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Conditionally Minimax Algorithm for Nonlinear System State Estimation

A. R. Pankov and A. V. Bosov

Abstract—In this note, a method of conditionally minimax nonlinear filtering (CMNF) of processes in nonlinear stochastic discrete-time controlled systems is proposed. The CMNF is derived by means of local nonparametric optimization of the filtering process given the class of admissible filters. Sufficient conditions for the existence of the CMNF are considered, and the properties of CMNF estimates are investigated. Results of the CMNF application to control and identification problems are presented.

I. INTRODUCTION

The problem of nonlinear system state estimation is a very important part of the general control problem for systems with incomplete data. It is well known that the nonlinear filtering problem involves a stochastic functional equation for the conditional distribution of the system state given all observations [1], [2]. So, the estimation problem in the general case is an infinite-dimensional one, and its exact solution is practically impossible.

Recently, various authors tried to determine some properties of the nonlinear system that, when fulfilled, provide finite dimensionality of the optimal estimator [3]–[8]. Investigations show that the situation when the optimal nonlinear filter is finite dimensional is not typical, and even in some very simple cases may not be the case [9]. Hence, we support the idea of obtaining the finite-dimensional estimators by means of appropriate approximations of the optimal nonlinear infinite-dimensional estimation algorithms. Some of these approximations are well known and widely used, such as the linearized and the extended Kalman filters (EKF), the second order filter, and many others [8], [10]. Their practical utilization shows that they, mostly being only suboptimal, have some undesirable properties, e.g., to provide biased and even divergent estimates [10], [11]. So, it is important to continue the efforts to obtain the nonlinear filters which are finite dimensional and provide estimates with improved properties.

In this note, we present a general approach for finite-dimensional nonlinear filtering. It is based on the idea of local conditional nonparametric optimization of the filter structure given the class

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of admissible filters. Locally optimal filters were considered by Pugachev [12], [13], who derived the conditionally optimal filter by parametric optimization of a nonlinear filter. To realize the last method, one needs to know the joint characteristic function of the system state and the filter estimate. This function can be obtained by solving the special nonrandom functional equation, which is rather complicated, and it takes much effort to obtain the desired solution, especially for the multidimensional case. Using the idea of conditionally optimal filtering, we derive a conditionally minimax nonlinear filter (CMNF), which can be determined by *a priori* computer modeling [14], [15].

To test the properties of the obtained algorithm we compare the results of the application of the CMNF, of the optimal nonlinear filter (ONF), and of the EKF to control and identification problems. Some simulation results are presented in this note.

II. CMNF ALGORITHM

Assume the following notations: $E\{x\}$ is the expectation of x ; $\text{cov}\{x, y\} = E\{(x - E\{x\})(y - E\{y\})^T\}$ is the covariance of x and y ; $P(m; S)$ is a set of random vectors x (probability distributions \mathcal{F}_x) with $E\{x\} = m$, $\text{cov}\{x, x\} = S$; A^+ is the pseudoinverse of A ; $A \geq 0$ if $A = A^T$ and is positive semidefinite; $A \leq B$ if $B - A \geq 0$; $\|x\|_W = (x^T W x)^{1/2}$ for some weight matrix $W \geq 0$; $\|x\| = \|x\|_I$ for $W = I$; and $\text{col}(x_1, \dots, x_n) = \{x_1^T, \dots, x_n^T\}^T$; $g^* \in \arg\min_{g \in G} f(g)$ if $f(g^*) = \inf_{g \in G} f(g)$.

Consider the following discrete-time dynamic model

$$\begin{cases} y_n = \varphi_n(y_{n-1}, u_n, w_n), & n = 1, 2, \dots; \quad y_0 = \eta, \\ z_n = \psi_n(y_n, v_n) \end{cases} \quad (1)$$

where $y_n \in \mathbb{R}^p$ is a state vector; $\{w_n\}$, $\{v_n\}$ are the independent white noises, $w_n \in \mathbb{R}^q$, $v_n \in \mathbb{R}^r$; $\eta \in \mathbb{R}^p$ is independent of $\{w_n\}$, $\{v_n\}$ random vector; $z_n \in \mathbb{R}^k$ is a vector of observations; $\varphi_n(y, u, w)$ and $\psi_n(y, v)$ are the known nonlinear functions; and $u_n = u_n(Z^{n-1}) \in \mathbb{R}^i$ is some feedback control, where $Z^n = \text{col}(z_n, \dots, z_1)$.

Let us consider the CMNF for the process $\{y_n\}$ given Z^n . Let $\xi_n(y, u)$ and $\zeta_n(y, z)$ be some fixed nonlinear vector functions (the choice of these functions is considered in Section IV), and \hat{y}_{n-1} be the CMNF-estimate of y_{n-1} given Z^{n-1} . Our objective is to obtain \bar{y}_n using \hat{y}_{n-1} , z_n and the structural functions $\xi_n(\cdot)$, $\zeta_n(\cdot)$ of CMNF introduced above. Let $\bar{y}_n = \xi_n(\hat{y}_{n-1}, u_n) \in \mathbb{R}^j$ be the basic prediction for y_n given Z^{n-1} . The conditionally minimax prediction $\tilde{y}_n \in \mathbb{R}^p$ for y_n is

$$\tilde{y}_n = \alpha_n^*(\bar{y}_n) \quad (2)$$

where

$$\alpha_n^*(\cdot) \in \arg\min_{\alpha(\cdot) \in A} \max_{X \in P(m_X, S_X)} E\{\|\alpha(\bar{y}_n)\|_W^2\} \quad (3)$$

and $X = \text{col}(y_n, \bar{y}_n) \in P(m_X, S_X)$, $A = \{\alpha(\cdot): E\{\|\alpha(\bar{y}_n)\|_W^2\} < \infty\}$, i.e., A is a set of functions $\alpha(\cdot): \mathbb{R}^j \rightarrow \mathbb{R}^p$ such that $E\{\|\alpha(\bar{y}_n)\|_W^2\}$ is finite.

Remark: From (2), (3) it follows that \tilde{y}_n is the best, in a minimax sense, estimate of y_n given the "observation" \bar{y}_n , and it is obtained under the assumption that the only known characteristics of $X = \text{col}(y_n, \bar{y}_n)$ are m_X and S_X . Note that the mean-square optimal estimate of y_n given \bar{y}_n is $y_n^* = E\{y_n | \bar{y}_n\}$ and to calculate y_n^* we need to derive the exact distribution of X . The last problem is equivalent to the initial optimal filtering one.

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Tuning, Multitone Instabilities, and Intrinsic Differences in Robustness of Adaptive Control Systems

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Abstract—An example is presented of an adaptive system that becomes unstable when a multitone reference input is applied, although it is stable when the spectral components are applied separately. It is further shown that another scheme, based on an input error instead of an output error in the adaptation, does not exhibit the same instability mechanism. The technique of averaging is used to analyze this phenomenon. The equilibrium points of the averaged system, called the tuned values, are investigated. Results are given on the number of tuned values, their locations, and their stability. The analysis is not only valid for a specific plant, but for a large class of systems.

I. INTRODUCTION

Robust adaptive control has been a major area of research in recent years. Indeed, the ability of a control system to maintain stability in the presence of unmodeled dynamics and measurement noise is a critical consideration in practical applications. In [6] and [14], it was shown that adaptive algorithms discussed in the literature at the time could be destabilized by relatively minor perturbations. A significant research interest followed to improve the robustness of adaptive control systems. Modifications of the update laws were developed such as the deadzone, relative deadzone, and leakage (see, e.g., the survey of [12]). These modifications were primarily designed to prevent the slow drift instability that occurs when inputs are not sufficiently rich and outputs are perturbed by measurement noise.

While the update law modifications are essential to the robustness of adaptive systems, other factors are also important. We concentrate here on the choice of adaptation mechanism, and show that the choice of error equation can lead to significant differences in robustness properties. The starting point of the paper is a new example, based on the example of [14]. A case of *multitone instability* is presented, where the adaptive system becomes unstable for a sum of sinusoids, even though it is stable for the spectral components applied separately. To study the example, we apply averaging theory, and study the equilibrium points of the averaged system, called the *tuned values*. We also analyze another scheme, based on an input error. We show that the adaptive system does not become unstable with the same multitone input, and we support, with the averaging analysis, the claim that the scheme is *intrinsically more robust* to unmodeled dynamics. Further, we explore an interesting *tuning property* of the algorithm.

Throughout the paper, we denote by $\hat{u}(s)$ the Laplace transform of a signal $u(t)$. $\hat{P}(s)$ denotes the transfer function of a linear time-invariant operator and $\hat{P}[u(t)]$ denotes the output of the operator with input $u(t)$. $\hat{P}[c_0 r]$ denotes the output of $\hat{P}(s)$ with input $c_0(t) \cdot r(t)$ (multiplied in the time domain). $\text{Re}[\cdot]$ and $\text{Im}[\cdot]$ denote the real and imaginary part of a complex number, respectively. OE is an abbreviation for output error and IE for input error.

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II. THE ROHRS EXAMPLES AND MULTITONE INSTABILITIES

We start from an example due to Rohrs *et al.* [14]. A model reference adaptive controller is designed assuming that the plant is a first-order, linear time-invariant system

$$\hat{P}(s) = \frac{k_p}{s + a_p} = \frac{\hat{y}_p(s)}{\hat{u}(s)} \quad (2.1)$$

where $k_p > 0$ and a_p are unknown. Nominally, $k_p = 2$ and $a_p = 1$. The control law is given by

$$\hat{u}(t) = c_0(t)r(t) + d_0(t)y_p(t). \quad (2.2)$$

For some nominal values of the adaptive parameters $c_0^* = k_m k_p^{-1}$, $d_0^* = (a_p - a_m)k_p^{-1}$, the closed-loop transfer function is equal to the transfer function of the reference model

$$\hat{M}(s) = \frac{k_m}{s + a_m} = \frac{3}{s + 3} = \frac{\hat{y}_m(s)}{\hat{r}(s)}. \quad (2.3)$$

The adaptive rules for $c_0(t)$, $d_0(t)$ are given by

$$\dot{c}_0 = -g c_0 r, \quad \dot{d}_0 = -g c_0 y_p \quad (2.4)$$

where $g > 0$ is called the adaptation gain. A Lyapunov analysis leads to the conclusion (cf. [11], [15]) that all the states of the adaptive system are bounded and that $c_0(t) \rightarrow 0$ as $t \rightarrow \infty$.

Rohrs *et al.* [14] observed that this adaptive system could be easily destabilized with the addition of noise and unmodeled dynamics. The unmodeled dynamics under consideration were of the form of multiplicative linear dynamics, so that the true plant was given by

$$\hat{P}(s) = \frac{2}{s + 1} \frac{229}{s^2 + 30s + 229} \quad (2.5)$$

instead of (2.1). A consideration was that the added poles were well-damped, stable, and far in the left-half plane, i.e., that the unmodeled dynamics were "mild" by classical standards. The instabilities described by Rohrs *et al.* were studied in detail by Astrom [2] (a full discussion is also available in [15]). A main contribution of the examples was to show what can go wrong when insufficient or improper excitation is provided. However, it has often been deduced from these examples that the more excitation there is in the desirable frequency range, the better things are. The reality is more complex. Indeed, consider the adaptive system with the plant (2.5) and no measurement noise. For a single tone at $\omega = 1$ rad/s, simulations show that the adaptive system is stable, and $c_0(t) \rightarrow 1.69$, $d_0(t) \rightarrow -1.26$. For $\omega = 10$ rad/s, the adaptive system is also stable, and $c_0(t) \rightarrow 1.04$, $d_0(t) \rightarrow -7.31$. However, if we let $r(t)$ be a multitone signal

$$r(t) = \sin(t) + 0.3 \sin(10t) \quad (2.6)$$

the adaptive system becomes unstable. The parameter responses for the single tone and multitone signals are shown in Fig. 1. The initial conditions used in the simulations are those of [14], that is, all zero except $c_0(0) = 1.14$, $d_0(0) = -0.65$. The adaptation gain is $g = 1$.

Note that there is plenty of excitation in the reference input signal, and most of it is in the desirable frequency range. In addition, there is no measurement noise. Therefore, the instability cannot be explained by the same arguments as for the Rohrs examples. The example also

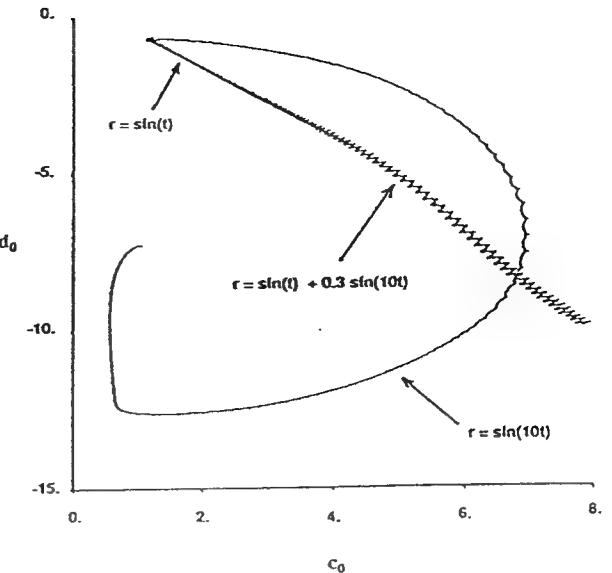


Fig. 1. Multitone instabilities, $d_0(c_0)$.

contradicts the notion of "good" and "bad" frequencies. For simpler adaptive systems (cf. the Riedle and Kokotovic example discussed in Section V), some frequencies cause instabilities, others do not, but their effect is cumulative. As long as there are more "good" frequencies than "bad" ones, the adaptive system remains stable. Here, the addition of a $\sin(t)$ signal to $\sin(10t)$ (both of these leading separately to stable systems) leads to instability. This is a highly nonlinear effect, absent in linear control systems. We might have hoped for some form of convexity: that adding two sinusoids would lead to a behavior of the adaptive system intermediate between the responses for the separate components. Such is not the case.

III. INPUT ERROR VERSUS OUTPUT ERROR MODEL REFERENCE ADAPTIVE CONTROL

The model reference adaptive control problem can be solved using several approaches, including direct and indirect techniques. These approaches have similar properties, established analytically, although there are some differences in assumptions and in computations. There are two main approaches to the model reference adaptive control problem with a *direct* algorithm: the *output error (OE)* approach, and the *input error (IE)* approach. The OE approach is the one used in the Rohrs examples and is developed in [11]. The IE approach was introduced in [8] in discrete time, and is the basis of the scheme of [7]. It is discussed extensively in continuous time in [15]. For the example considered in Section II, the error driving the algorithm is given by

$$e_2 = c_0 y_p + d_0 \hat{M}[y_p] - \hat{M}[u] \quad (3.1)$$

with the adaptation law

$$\dot{c}_0 = -g e_2 y_p, \quad \dot{d}_0 = -g c_2 \hat{M}[y_p]. \quad (3.2)$$

The scheme is a simplified version of the general scheme discussed in [15] (in the notation of [15], the special choice of \hat{L} was made such that $\hat{L}\hat{M} = 1$). Also, two adjustments to the scheme that are needed to prove stability have been omitted: the normalization factor in the update law, and the projection of c_0 to prevent $c_0 = 0$.

Consider now the example of Section II, that is, the Rohrs example with a multitone input and with the input error scheme replacing the output error algorithm. Fig. 2 shows the responses of the adaptive

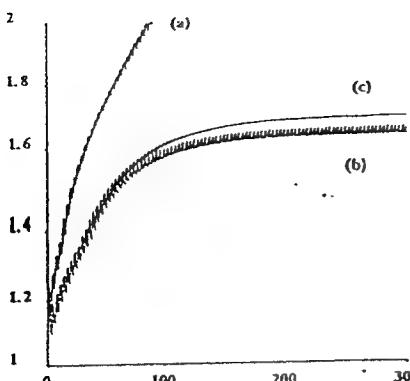


Fig. 2. $c_0(t)$. (a) OE, $r = \sin(t) + 0.3 \sin(10t)$, (b) IE, $r = \sin(t) + 0.3 \sin(10t)$, (c) IE, $r = \sin(t)$.

parameter c_0 for the single-tone input and for the multitone input. The response of d_0 is similar and is omitted for brevity. Again, the initial conditions are all zero, except $c_0(0) = 1.14$, $d_0(0) = -0.65$. The gain $g = 1$. The input error scheme is not destabilized by the multitone input. For single-tone inputs, the parameters converge to the same values as the output error scheme, i.e., the values that achieve model matching. In the presence of the multitone input, the parameters converge to values close to those obtained for the $\sin(t)$ input, a clearly desirable feature since this is the dominant component.

The averaging analysis of Sections V and VI will confirm that the different responses of the algorithms reflect intrinsic differences of robustness and not a peculiar choice of example. The analysis will also allow us to answer questions that naturally arise, such as: Toward which values do the parameters of the IE scheme converge in the presence of multitone inputs? Does the convergence depend on the initial conditions? Why is the OE scheme unstable?

IV. AVERAGING THEORY AND TUNED VALUES

Averaging is a method of analysis for differential equations of the form

$$\dot{x} = \epsilon f(t, x), \quad x(0) = x_0. \quad (4.1)$$

The main assumptions are that $\epsilon > 0$ is small and that the limit

$$f_{av}(x) = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau, x) d\tau \quad (4.2)$$

exists uniformly in t_0 and x . The *averaged system* is then defined by

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}), \quad x_{av}(0) = x_0. \quad (4.3)$$

Given some smoothness assumptions on f and f_{av} , it can be shown that, for ϵ sufficiently small, the trajectories $x(t)$ and $x_{av}(t)$ are arbitrarily close as $\epsilon \rightarrow 0$ on intervals $[0, T/\epsilon]$, where T is arbitrary. Further, if $x_{av} = x^+$ is an exponentially stable equilibrium point of the averaged system, the trajectories of the original system locally converge to a ball around x^+ , whose size shrinks to zero as $\epsilon \rightarrow 0$. The theory can also be extended to so-called *two-time scale* systems, of the form

$$\dot{x} = \epsilon f(t, x, y) \quad (4.4)$$

$$\dot{y} = A(x)y + h(t, x). \quad (4.5)$$

Equations (4.4)–(4.5) are especially natural in adaptive control, where x is the adaptive parameter vector and y is the vector of plant and controller states. Averaging has been applied in this context by several authors (among others, [1], [10], [15]). The assumption of ϵ small is equivalent to the assumption of slow adaptation, i.e., that the adaptive

parameters vary slowly compared to the other states of the system. The question of stability is then divided into two parts: the *inner-loop stability*, i.e., the stability of the closed-loop system with fixed controller parameters, or the stability of the matrix $A(x)$, and the *outer-loop stability*, i.e., the stability of the adaptation itself, or the stability of the nonlinear averaged system.

This paper concentrates on the question of outer-loop stability. An interesting example of the use of averaging analysis to guarantee inner-loop stability is found in the work of [5], in the context of adaptive echo cancellation. As for outer-loop stability, averaging allows us to make precise the notion of *tuned values*, first introduced in [9] and discussed in the context of averaging in [4]. Assuming that the average exists, the tuned values x^+ are defined as the equilibrium points of the averaged system, i.e., the solutions of $f_{av}(x_{av}^+) = 0$. A tuned value is said to be stable if the equilibrium is locally asymptotically stable.

V. AVERAGING ANALYSIS—FEEDFORWARD GAIN ALONE

In this section, we apply the averaging theory to the analysis of the adaptive schemes of Sections II and III, but with only the feedforward gain c_0 being updated. This intermediate step will be helpful to interpret the later results. While the analysis is restricted to the specific adaptive scheme with one parameter, there is considerable generality in the results, as the plant and reference model are taken to be arbitrary transfer functions. The adaptive system with one parameter c_0 was considered by Riedle and Kokotovic [13] and further studied in [3]. The equations for the adaptive systems are the same as before, except that $d_0 = 0$. For the stability and for the averaging analysis, the following assumptions are made.

(A1) The plant transfer function $\hat{P}(s)$ is stable. (A2) The input r is given by $r = \sum_{i=1}^n r_i \sin(\omega_i t)$ with $r_i \neq 0$.

Assumptions (A1) and (A2) are sufficient to guarantee the existence of the averaged system. Assumption (A2) can be slightly relaxed, but this form will simplify the presentation.

Proposition 5.1—Averaged Systems for the OE and IE Schemes: The averaged system for the output error scheme is given by

$$\dot{c}_{av} = -\frac{g}{2} \sum_{i=1}^n r_i^2 (\text{Re}[\hat{P}(j\omega_i)] c_{av} - \text{Re}[\hat{M}(j\omega_i)]) \quad (5.1)$$

and for the input error scheme

$$\dot{c}_{av} = -\frac{g}{2} c_{av}^2 \sum_{i=1}^n r_i^2 (|\hat{P}(j\omega_i)|^2 c_{av} - \text{Re}[\hat{M}(j\omega_i) \hat{P}^*(j\omega_i)]). \quad (5.2)$$

The proof is given in [3].

Proposition 5.2—Tuned Values for the OE Scheme: Assume that $\sum_{i=1}^n r_i^2 \text{Re}[\hat{P}(j\omega_i)] \neq 0$. Then, the averaged system for the output error scheme has a unique tuned value:

$$c_{av}^+ = \frac{\sum_{i=1}^n r_i^2 \text{Re}[\hat{M}(j\omega_i)]}{\sum_{i=1}^n r_i^2 \text{Re}[\hat{P}(j\omega_i)]} \quad (5.3)$$

which is stable if and only if $\sum_{i=1}^n r_i^2 \text{Re}[\hat{P}(j\omega_i)] > 0$.

Comments: Proposition 5.2 follows directly from Proposition 5.1 and is essentially the result of Riedle and Kokotovic [13]. The authors also showed, through simulations, that the stability criterion establishes a tight stability-instability boundary for the original adaptive system. The condition is satisfied if $\hat{P}(s)$ is strictly positive real, which requires $\text{Re}[\hat{P}(j\omega)] > 0$ for all $\omega \geq 0$. However, the condition is weaker in the sense that the transfer function must have a positive real part only at the frequencies of excitation and, further, some “undesirable” frequencies can be tolerated as long as the contributions of the frequencies in the positive region exceed those in the negative region.

Proposition 5.3—Tuned Values for the IE Scheme: Assume that $|\hat{P}(j\omega_i)| \neq 0$ for some i . Then, aside from $c_{av}^+ = 0$, the averaged system for the input error scheme has a unique tuned value given by

$$c_{av}^+ = \frac{\sum_{i=1}^n r_i^2 \operatorname{Re}[\hat{M}(j\omega_i)\hat{P}^*(j\omega_i)]}{\sum_{i=1}^n r_i^2 |\hat{P}(j\omega_i)|^2} \quad (5.4)$$

which is always stable. Further, c_{av}^+ minimizes the averaged squared error.

Comments: Proposition 5.3 follows directly from Proposition 5.1. While the output error scheme can be destabilized by a high-frequency input, such is not the case for the input error scheme. In fact, the surprising result is that no frequency or combination of frequencies can destabilize the scheme. The result is independent of the specific transfer function $\hat{P}(s)$ and, therefore, shows that the scheme is intrinsically more robust. Perhaps the most intriguing part of the results is the fact that the tuned value for the input error scheme minimizes the averaged squared error, although the output error does not appear explicitly in the adaptation law. It is often believed that the fact that the output error is used to update the parameter in the output error scheme is a guarantee that the error would be minimized. This is not true, however, as the scheme can, in fact, be destabilized, leading to unbounded errors. On the other hand, the input error scheme always converges to a neighborhood of the parameter value for which the error is minimized.

VI. AVERAGING ANALYSIS—FEEDFORWARD AND FEEDBACK GAINS

For the averaging analysis, we make the following assumptions.

(A1) Along the trajectories of the averaged system, the closed-loop transfer function $\hat{P}(s)/(1 - d_0 \hat{P}(s))$ is stable. (A2) The input r is given by $r = \sum_{i=1}^n r_i \sin(\omega_i t)$ with $r_i \neq 0$, $\omega_i \neq 0$. (A3) $\operatorname{Re}[\hat{M}(j\omega_i)] > 0$, $\operatorname{Im}[\hat{M}(j\omega_i)] < 0$ for all i .

Assumption (A1) is hard to guarantee *a priori* and is usually checked *a posteriori*: the stability of a tuned value will guarantee the local stability of the adaptive system if the tuned value corresponds to a stable inner loop. Assumption (A2) restricts the class of inputs so that the averaging analysis is possible. The conditions $r_i \neq 0$ and $\omega_i \neq 0$ imply that the input is sufficiently rich to guarantee exponential parameter convergence in the ideal case (no unmodeled dynamics). Assumption (A3) is not really a new assumption since $\hat{M}(s)$ is given by (2.3). However, the results are valid for arbitrary $\hat{M}(s)$ provided that the function satisfies (A3).

Theorem 6.1—Averaged Systems for the OE and IE Schemes: The averaged system for the output error scheme is given by

$$\begin{pmatrix} \dot{c}_{av} \\ \dot{d}_{av} \end{pmatrix} = -\frac{g}{2} C(c_{av}, d_{av}) \sum_{i=1}^n \frac{r_i^2}{\Delta_i(d_{av})} \begin{pmatrix} A_i \begin{pmatrix} c_{av} \\ d_{av} \end{pmatrix} - b_i \end{pmatrix} \quad (6.1)$$

where

$$A_i = \begin{pmatrix} \operatorname{Re}[\hat{P}(j\omega_i)] & \operatorname{Re}[\hat{P}(j\omega_i)\hat{M}(j\omega_i)] \\ |\hat{P}(j\omega_i)|^2 & |\hat{P}(j\omega_i)|^2 \operatorname{Re}[\hat{M}(j\omega_i)] \end{pmatrix} \quad (6.2)$$

$$b_i = \begin{pmatrix} \operatorname{Re}[\hat{M}(j\omega_i)] \\ \operatorname{Re}[\hat{P}(j\omega_i)\hat{M}^*(j\omega_i)] \end{pmatrix} \quad C(c_{av}, d_{av}) = \begin{pmatrix} 1 & -d_{av} \\ 0 & c_{av} \end{pmatrix} \quad (6.3)$$

$$\Delta_i(d_{av}) = (1 - d_{av} \operatorname{Re}[\hat{P}(j\omega_i)])^2 + (d_{av} \operatorname{Im}[\hat{P}(j\omega_i)])^2. \quad (6.4)$$

For the input error scheme, (6.1) also holds, but with

$$A_i = |\hat{P}(j\omega_i)|^2 \begin{pmatrix} 1 & \operatorname{Re}[\hat{M}(j\omega_i)] \\ \operatorname{Re}[\hat{M}(j\omega_i)] & |\hat{M}(j\omega_i)|^2 \end{pmatrix} \quad (6.5)$$

$$b_i = \begin{pmatrix} \operatorname{Re}[\hat{P}(j\omega_i)\hat{M}^*(j\omega_i)] \\ |\hat{M}(j\omega_i)|^2 \operatorname{Re}[\hat{P}(j\omega_i)] \end{pmatrix} \quad C(c_{av}, d_{av}) = \begin{pmatrix} c_{av}^2 & 0 \\ 0 & c_{av}^2 \end{pmatrix} \quad (6.6)$$

$$\Delta_i(d_{av}) = (1 - d_{av} \operatorname{Re}[\hat{P}(j\omega_i)])^2 + (d_{av} \operatorname{Im}[\hat{P}(j\omega_i)])^2. \quad (6.7)$$

Comments: The proof of the theorem is in the Appendix. The averaged systems have been written in a common and compact form that exhibits the similarities with the one-parameter case. However, the extension from one to two parameters has considerably increased the complexity. The main reason is that feedback has made the system nonlinear. The term $\Delta_i(d_{av})$ is a polynomial of degree 2 that is always positive and is the square of the closed-loop characteristic polynomial evaluated at $s = j\omega_i$. A similar term is found in the n -parameter case without unmodeled dynamics [15]. The matrix $C(c_{av}, d_{av})$ is another nonlinear term, but since it is nonsingular for $c_{av} \neq 0$, it has no effect on the location of the tuned values.

For the rest, most of the difference between the OE scheme and the IE scheme is in the matrices A_i . In the IE scheme, the matrices A_i are symmetric, and they are positive definite as long as $|\hat{P}(j\omega_i)| \neq 0$ (semi-definite otherwise). This is reminiscent of Section V, where the coefficient of c_{av} was $|\hat{P}(j\omega_i)|^2$. On the other hand, the coefficient for the OE scheme in the scalar case was $\operatorname{Re}[\hat{P}(j\omega_i)]$, which could be negative in the presence of unmodeled dynamics. In the two-parameter case of the OE scheme, the matrices A_i are generally neither symmetric nor positive definite. We will see the implications of these properties in the next theorems.

Theorem 6.2—Tuned Values for the OE Scheme: For $n = 1$: if $|\hat{P}(j\omega_1)| \neq 0$, the averaged system for the output error scheme has a unique tuned value:

$$\begin{pmatrix} c_{av}^+ \\ d_{av}^+ \end{pmatrix} = \begin{pmatrix} \operatorname{Im}[\hat{P}(j\omega_1)] |\hat{M}(j\omega_1)|^2 / \operatorname{Im}[\hat{M}(j\omega_1)] |\hat{P}(j\omega_1)|^2 \\ \operatorname{Im}[\hat{P}^*(j\omega_1)\hat{M}(j\omega_1)] / \operatorname{Im}[\hat{M}(j\omega_1)] |\hat{P}(j\omega_1)|^2 \end{pmatrix} \quad (6.8)$$

which is stable if $\operatorname{Im}[\hat{P}(j\omega_1)] < 0$. The tuned value is such that the closed-loop transfer function matches the reference model transfer function at ω_1 .

For arbitrary n : the averaged system for the output error scheme has at most $4n - 3$ tuned values (aside from a possible tuned value with $c_{av}^+ = 0$).

Comments: The proof of the theorem is left to the Appendix. The results are, unfortunately, much more limited than in the case of a feedforward gain alone. Of course, we should not expect too much, given the weak assumptions made on the plant transfer function. However, we will also show by example that the tuned values cannot be bounded, that there may exist one, several, or none at all, and that they may all be unstable.

In the case of a single sinusoid, the tuned value exists, it is unique, and it is the value that achieves model matching. This is consistent with simulations and the observations of Astrom [2]. The tuned value becomes unstable when $\operatorname{Im}[\hat{P}(j\omega_1)] > 0$. However, this happens at the same time as when the tuned value corresponds to an unstable inner loop. In other words, inner- and outer-loop instabilities occur at the same time. This is different from the Riedle and Kokotovic example, where inner-loop stability was always guaranteed.

Theorem 6.3—Tuned Values for the IE Scheme: For $n = 1$: if $|\hat{P}(j\omega_1)| \neq 0$, the averaged system for the input error scheme has a unique tuned value (aside from $c_{av}^+ = 0$), which is the same as for the output error scheme, and which is always stable.

For arbitrary n : there exists at least one tuned value and at most $4n - 3$. The tuned values are bounded by

$$\left| \begin{pmatrix} c_{av}^+ \\ d_{av}^+ \end{pmatrix} \right| \leq \frac{\sum_{i=1}^n |b_i|}{(\min_i \lambda_{\min}(A_i))}. \quad (6.9)$$

Comments: The proof of the theorem is left to the Appendix. The results for the input error scheme are more significant than for the output error scheme. We are guaranteed that there exists at least one tuned value, and that all the tuned values are bounded. Still, the

results are less significant than in the case of the feedforward gain alone, except when there is a single sinusoid. In that case, the results are similar to the case of the feedback gain alone, guaranteeing not only the uniqueness, but also the stability of the tuned value. Again, it should be remembered that this guarantees the stability of the adaptive system only if the tuned value corresponds to a stable inner loop.

The main generic difference between the OE and IE schemes is that the tuned values cannot be bounded in the case of the OE scheme. This lack of boundedness can be traced to the fact that, although the matrices A_i in (6.1) are summed with positive weights, the resulting matrix may be singular in the OE scheme. On the other hand, in the case of the IE scheme, the matrices are symmetric positive definite so that their weighted sum will always be positive definite. This fact can be extended to adaptive systems with the number of parameters greater than two, and is not limited to the two-parameter case.

VII. ANALYSIS OF THE MULTITONE INSTABILITIES

We now return to the example of Section II, with $r = r_1 \sin(t) + r_2 \sin(10t)$, and use the results of Section VI. We observed that the adaptive system was unstable for $r_1 = 1, r_2 = 0.3$. However, the system was stable for $r_1 = 1, r_2 = 0$ and $r_1 = 0, r_2 = 1$. To gain insight into this phenomenon, the tuned values were calculated for various values of r_1, r_2 . Within the assumptions of averaging (sufficiently small adaptation gain), only the ratio of r_1 over r_2 is important. Therefore, we let $r_1 = 1 - r_2$. As is shown in the Appendix, the tuned values can be calculated by eliminating c_{av}^+ and solving a fifth-order equation for d_{av}^+ . Fig. 3 shows the locus of d_{av}^+ as r_2 is varied from 0 to 1 and $r_1 = 1 - r_2$. S denotes a stable tuned value and U an unstable one. The cases when $r_2 = 0$ and $r_2 = 1$ correspond to single tones, for which we confirm that a unique and stable tuned value exists. However, in the intermediate region, the transition is far from being smooth. In particular, the simulation of Section II corresponds to a case where a unique *unstable* tuned value exists, which explains the instability of the original adaptive system. For other ratios, there may exist up to three tuned values (the maximum of five is not reached), or none at all. Even though the absence of tuned values occurs only for specific values of r_2 (e.g., $r_2 = 0.16$), the tuned values become unbounded as the ratio is approached, proof of the fact that *no bound* can be found for the tuned values of the output error scheme.

The locus for the input error scheme is shown in Fig. 4. Again, the cases when $r_2 = 0$ or $r_2 = 1$ correspond to single tones, for which unique and stable tuned values are known to exist, identical to those of the output error scheme. The transition between the two is far smoother than previously, however, and the tuned values always remain bounded. Further, there always exists a tuned value, and there also always exists a *stable* tuned value (this is not guaranteed by Theorem 6.3).

Fig. 4 also gives special justification to the name of tuned value. For a large range of amplitudes of the additional sinusoid at $\omega = 10$ rad/s, the tuned value remains close to its original value for a single tone at $\omega = 1$ rad/s. The adaptive system *tunes itself* to the low-frequency component. When the component at $\omega = 10$ rad/s is dominant, the adaptive system *tunes itself* to the high frequency by adjusting its parameter close to the value of the parameter obtained for the high frequency alone.

There is also a range of amplitudes where the adaptive system may tune itself either to the low-frequency or to the high-frequency component. Figs. 5 and 6 show the steady-state output error responses for $d_0(0) = -3$ and $d_0(0) = -4$, respectively. The parameters $r_1 = 0.24, r_2 = 0.76$, and $g = 10$. In the first case, the adaptive parameter d_0 converges to a value close to the value for a single

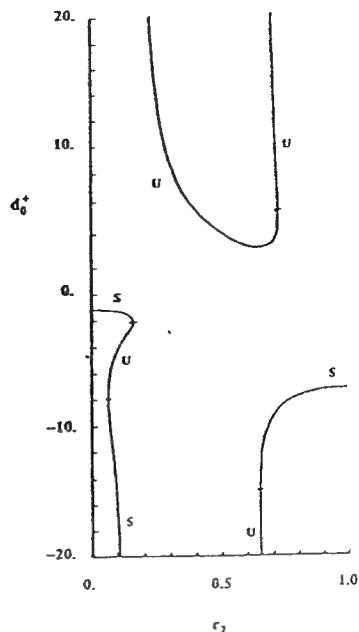


Fig. 3. Locus of tuned values, output error scheme.

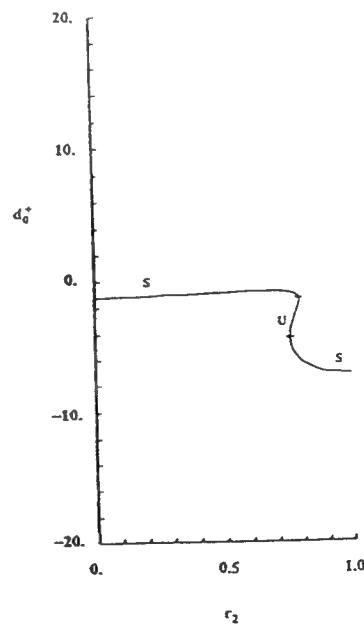
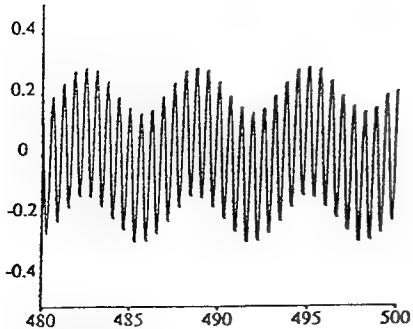
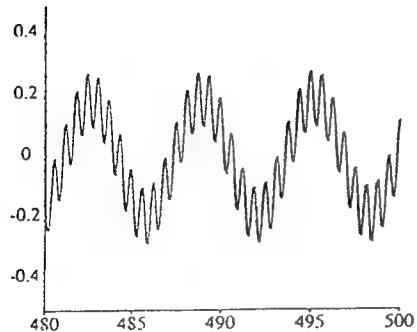


Fig. 4. Locus of tuned values, input error scheme.

tone at $\omega = 1$ rad/s, and the residual error consists mostly of the component at $\omega = 10$ rad/s. In the second case, the adaptive parameter converges to a value close to the value for a single tone at $\omega = 10$ rad/s, and the residual error consists mostly of the component at $\omega = 1$ rad/s. The adaptive system either tunes itself to the low-frequency or to the high-frequency component, depending on initial conditions. In both cases, the residual output error has a comparable magnitude. Although the existence of a unique tuned value might have been found desirable *a priori*, we see that the higher number is not necessarily contradictory to the correct behavior of the adaptive scheme.

Before completing this discussion, we may reflect on the reasons why the two model reference adaptive control schemes lead to totally different responses to multitone inputs. Aside from the results of

Fig. 5. $c_0(t)$ for $d_0(0) = -3$.Fig. 6. $c_0(t)$ for $d_0(0) = -4$.

the averaging analysis, we would like to propose the following explanation. In Section V, we reviewed the one-parameter case. The conclusion was that the scheme was stable for single sinusoids provided that $\text{Re}[\hat{P}(j\omega_1)] > 0$. When several sinusoids were present, their effect was cumulative, i.e., the stability was determined by the weighted sum of $\text{Re}[\hat{P}(j\omega_i)]$. In the case of the two-parameter example, the stability in the presence of a single sinusoidal input is also guaranteed for sufficiently low frequencies. However, the condition is that $\text{Im}[\hat{P}(j\omega_1)] < 0$. Note that, in the example, $\text{Re}[\hat{P}(j\omega)] > 0$ for $\omega < 2.7$ rad/s, but $\text{Im}[\hat{P}(j\omega)] < 0$ for $\omega < 16.1$ rad/s. Why is the range of allowable frequencies larger in the two-parameter case than in the one-parameter case? It turns out to be because the feedback gain d_0 makes the real part of the *closed-loop* transfer function positive for a larger range of frequencies. Indeed, it can be checked that $\text{Im}[\hat{P}(j\omega_1)] < 0$ is also the condition such that the tuned value given by Theorem 6.2 makes the transfer function from variations δc_0 , δd_0 to variations δc_0 strictly positive real.

In the presence of multitone inputs, model matching can only be approximately achieved. If the low frequency component is dominant, the adaptive system may tune itself to the low frequency. A high-frequency residual error will then be present. However, the real part of the closed-loop transfer function is not positive at that higher frequency. Simulations and calculations have told us that this perturbation can easily destabilize the output error scheme. On the other hand, the input error scheme does not require strictly positive real conditions for stability in the case of single-tone inputs. We found that it is not destabilized by the multitone input.

VIII. CONCLUSIONS

In this paper, we observed some highly nonlinear phenomena in adaptive systems. In particular, an example was presented of an

adaptive system with unmodeled dynamics which was stable for sinusoidal reference inputs, but became unstable for the sum of two such signals. Other situations were found where several regions of attraction of the adaptive parameters existed. Their number depended on the location of the spectral components and on their respective amplitudes.

The technique of averaging was used to analyze these effects, and demonstrated its power for the study of the dynamic behavior of adaptive systems. The concept of tuned value as an equilibrium point of the averaged system was found to be particularly useful. It is hard to imagine how one could have made sense of the adaptive system responses to various multitone inputs without the peculiar locus of tuned values shown in Fig. 3. Intuitive explanations of the multitone instabilities were also provided. We argued that the minimization of the tracking error in the presence of unmodeled dynamics led to conflicting requirements which, in turn, resulted either in tuning or instabilities. The effect of the choice of error equation was found to be particularly important. We showed that the input error scheme was more robust, and this claim was not only supported by simulations on an example, but also by an analysis that did not depend on specific unmodeled dynamics.

The observations of this paper are not contradictory to other research results in robust adaptive control emphasizing update law modifications. These modifications should be used to increase robustness and prevent parameter drift in the presence of noise and inputs that are not sufficiently rich. Also, the effect of input disturbances and measurement noise was not addressed in this work, and a careful study might lead to interesting conclusions.

APPENDIX

Proof of Theorem 6.1: We prove the result for the output error scheme only. The proof for the input error scheme follows along similar lines. Assuming that the parameters c_0 and d_0 are fixed, the output is given by $y_p = (c_0 \hat{P} / 1 - d_0 \hat{P})[r]$, and therefore,

$$\begin{aligned} (y_p - y_m)y_p &= \frac{c_0 \hat{P}}{1 - d_0 \hat{P}}[r] \cdot \frac{c_0 \hat{P}}{1 - d_0 \hat{P}}[r] - \frac{c_0 \hat{P}}{1 - d_0 \hat{P}}[r] \cdot \hat{M}[r] \\ &= c_0^2 \frac{\hat{P}}{1 - d_0 \hat{P}}[r] \cdot \frac{\hat{P}}{1 - d_0 \hat{P}}[r] - c_0 \frac{\hat{P}}{1 - d_0 \hat{P}}[r] \\ &\quad \cdot \frac{\hat{M}}{1 - d_0 \hat{P}}[r] + c_0 d_0 \frac{\hat{P}}{1 - d_0 \hat{P}}[r] \cdot \frac{\hat{P} \hat{M}}{1 - d_0 \hat{P}}[r]. \end{aligned} \quad (\text{A1.1})$$

By assumption (A2), the average of (A1.1) is therefore given by

$$\begin{aligned} &\sum_{i=1}^n \frac{r_i^2}{2} c_0 \frac{1}{|1 - d_0 \hat{P}(j\omega_i)|^2} \\ &\cdot (c_0 |\hat{P}(j\omega_i)|^2 - \text{Re}[\hat{P}(j\omega_i) \hat{M}^*(j\omega_i)]) \\ &+ d_0 |\hat{P}(j\omega_i)|^2 \text{Re}[\hat{M}(j\omega_i)]). \end{aligned} \quad (\text{A1.2})$$

The average of $(y_p - y_m)r$ is similarly obtained and, by making the appropriate definitions, the results of the theorem follow.

Proof of Theorem 6.2: For $n = 1$: assume first that $C(c_{av}, d_{av})$ is nonsingular. Under the conditions of the theorem, A_1 is nonsingular, so that the tuned value is unique and given by $A_1^{-1}b_1$. It can also be checked that the tuned value solves $c_{av}^+ \hat{P}[j\omega_1] / 1 - d_{av}^+ \hat{P}[j\omega_1] = \hat{M}(j\omega_1)$, i.e., that the tuned value corresponds to the matching of the reference model by the closed-loop transfer function. The stability of the averaged system around the tuned value is determined by the

linearized system and, therefore, by the stability of the matrix

$$\begin{aligned} & -g \frac{r_1^2}{2} C(c_{av}, d_{av}) A_1 \\ & = -g \frac{r_1^2}{2} \frac{\text{Im}[\hat{P}(j\omega_1)]}{\text{Im}[\hat{M}(j\omega_1)]} \\ & \cdot \begin{pmatrix} \text{Re}[\hat{M}(j\omega_1)] & (\text{Re}[\hat{M}(j\omega_1)])^2 - (\text{Im}[\hat{M}(j\omega_1)])^2 \\ |\hat{M}(j\omega_1)|^2 & |\hat{M}(j\omega_1)|^2 \text{Re}[\hat{M}(j\omega_1)] \end{pmatrix}. \end{aligned} \quad (\text{A2.1})$$

Again, it is easy to check that this system is stable under the conditions of the theorem. Now, let us return to the assumption that $C(c_{av}, d_{av})$ is nonsingular. C is singular if and only if $c_{av} = 0$. Assuming that a tuned value with $c_{av}^+ = 0$ exists, it would have to satisfy

$$\begin{aligned} d_{av}^{+2} |\hat{P}(j\omega_1)|^2 \text{Re}[\hat{M}(j\omega_1)] - d_{av}^+ \text{Re}[\hat{P}(j\omega_1) \hat{M}(j\omega_1)] \\ - d_{av}^+ \text{Re}[\hat{P}(j\omega_1) \hat{M}^*(j\omega_1)] \\ + \text{Re}[\hat{M}(j\omega_1)] = 0 \end{aligned} \quad (\text{A2.2})$$

and therefore $d_{av}^{+2} |\hat{P}(j\omega_1)|^2 - 2d_{av}^+ \text{Re}[\hat{P}(j\omega_1)] + 1 = 0$. However, this equation has no real solution if $\text{Im}[\hat{P}(j\omega_1)] \neq 0$.

For arbitrary n , the equation for the tuned value is quite more complicated. If $c_{av}^+ \neq 0$, $C(c_{av}, d_{av})$ is nonsingular. Take $n = 2$ for simplicity. Then, c_{av}^+, d_{av}^+ satisfy

$$r_1^2 \Delta_2(d_{av}^+) \left(A_1 \begin{pmatrix} c_{av}^+ \\ d_{av}^+ \end{pmatrix} - b_1 \right) + r_2^2 \Delta_1(d_{av}^+) \left(A_2 \begin{pmatrix} c_{av}^+ \\ d_{av}^+ \end{pmatrix} - b_2 \right) = 0. \quad (\text{A2.3})$$

This is a set of two polynomial equations in c_{av}^+, d_{av}^+ . The maximum power of d_{av}^+ is 3 and of c_{av}^+ is 1. The coefficient of c_{av}^+ is a polynomial in d_{av}^+ of degree 2. Therefore, c_{av}^+ can be eliminated to yield an equation of degree 5 for d_{av}^+ , which has at most five real solutions. c_{av}^+ is then determined uniquely from d_{av}^+ . The procedure is easily extended for arbitrary n , indicating that there are at most $4n - 3$ tuned values.

Proof of Theorem 6.3: The proof is similar to the proof of Theorem 6.2. The matrix of the linearized system is now

$$\begin{aligned} -g \frac{r_1^2}{2} C(c_{av}, d_{av}) A_1 = -g \frac{r_1^2}{2} c_{av}^2 |\hat{P}(j\omega_1)|^2 \\ \cdot \begin{pmatrix} 1 & \text{Re}[\hat{M}(j\omega_1)] \\ \text{Re}[\hat{M}(j\omega_1)] & |\hat{M}|^2 \end{pmatrix} \end{aligned} \quad (\text{A3.1})$$

which is always stable. The fact that the matrices A_i are symmetric and positive definite implies the additional results. In particular, the equation for d_{av}^+ is again a polynomial of order $4n - 3$, but its leading coefficient is guaranteed to be nonzero. Since the order is odd, there must exist at least one real tuned value. The bound on the tuned values also follows from the positive definiteness of the matrices A_i .

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The Effect of Time Delay and Discrete Control on the Contact Stability of Simple Position Controllers

John Fiala and Ronald Lumia

Abstract—By analysis of the driving-point admittance, it is shown how time delays and discrete control can create instabilities for a simple position controller in contact with the environment. The lowest frequency of contact instability due to time delay or sampling is determined analytically. It is shown how mechanical compliance between the motor and point of contact can eliminate these instabilities. To achieve the best relative stability when contacting arbitrary environments, the mechanical/control design of manipulators should maintain a critical relationship between the frequency of the compliant mode and a frequency associated with contact instability.

I. INTRODUCTION

The simple proportional-derivative (PD) controllers used for controlling most robots show a remarkable robustness in a number of tasks, including those which involve contact with the environment. Recently, some authors have noted that the time delays and sampling in these controllers should have a detrimental effect on stability during contact with certain environments. Goldenberg and Clark [1] describe

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Identification with Modeling Uncertainty and Reconfigurable Control

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Abstract

The problem of obtaining reliable estimates of uncertainty in the parameters identified through a least-squares algorithm is discussed. Estimates based on a stochastic analysis, an analysis assuming bounded noise, and a sensitivity analysis are reviewed. The results are compared and illustrated using experimental data obtained on a DC motor. The need for methods of estimation of uncertainty is justified in the context of adaptive control, where robustness and transient performance are critical. In particular, the application to reconfigurable flight control is considered. Design tradeoffs for this application are discussed in detail and illustrated through simulations using two aircraft models.

1. Introduction

There has been a considerable interest in recent years in new methods of identification that account for modeling uncertainty as well as measurement noise. Reasons for this interest include:

- traditional methods of identification do not provide the kind of information that is used in robust control system design, namely measures of nonparametric dynamic uncertainty;
- by not accounting for such uncertainty, traditional methods often provide overly optimistic estimates of the parametric uncertainty;
- stochastic measures of parametric uncertainty are probabilistic, whereas robustness measures assume hard bounds on transfer function models.

Recent papers that address various parts of this problem include [1]-[9]. The adaptive control community is particularly interested in developing robust adaptive controllers using the simple paradigm that robust adaptive control equals robust identification plus robust control. Indeed, there is evidence that the robustness and transient performance properties of existing algorithms need to be better understood. In a recent paper [10], we found that the transient response of certain adaptive algorithms exhibited repeated large peaks. A similar phenomenon called burst phenomenon is known to occur in adaptive systems with measurement noise but, in this case, the responses were observed in systems without any noise or unmodeled dynamics. In [11], we discussed the case of an adaptive system which became unstable when the reference input was composed of the sum of two sinusoids, even though it was stable when excited by a single sinusoidal component. Such examples demonstrate

the complexity of the issues of transient performance and robustness in adaptive systems, and justify an interest for alternate approaches.

Traditional adaptive control systems are based on the certainty equivalence principle, so that a recursive identifier is combined with a control algorithm which uses the estimated parameters as if they were perfect estimates. New adaptive algorithms would not only use the estimates of the parameters, but also the estimated uncertainty affecting these parameters (see Fig. 1).

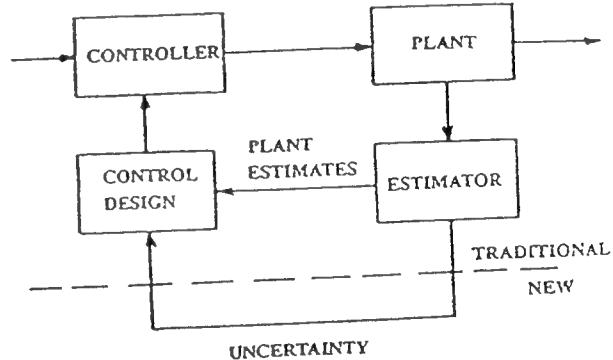


Figure 1: Traditional and New Adaptive Control Structures

By being more cautious when necessary, one may expect that such new methods will exhibit superior transient performance and robustness properties.

A special area of application of robust adaptive control is that of reconfigurable control, which we define as the control of multi-variable systems in the presence of drastic changes in dynamics due, for example, to actuator failures. In such cases, the structure of the controller may have to be altered (as opposed to fine-tuning of the parameters). The U.S. Air Force has been interested in reconfigurable control because of the potential benefit for increased survivability and reduced need for hardware redundancy [12], [13]. In flight control, it is difficult to provide fault tolerance using fault detection and identification because of the large number of possible failures and the large number of flight conditions. Methods based on automatic system identification and control reconfiguration are therefore attractive, but there is a critical need for an adaptive system that is both robust to uncertainties and well-behaved in the transient regime.

2. Identification with Modeling Uncertainty

Least-Squares Algorithm: Consider the linear regression

$$y(t) = w^T(t) \theta^* + d(t) \quad (1)$$

where $y(t)$ is a measured signal, $w(t)$ is a vector of measured signals, $d(t)$ is an unknown signal, and θ^* is a vector of

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unknown parameters. The least-squares algorithm provides the estimate θ_{LS} that minimizes the error

$$R_e(\theta) = \sum_{t=1}^N (y(t) - w^T(t) \theta)^2 \quad (2)$$

and is given by

$$\theta_{LS} = R_w^{-1} \cdot R_{wy} \quad (3)$$

where

$$R_w = \sum_{t=1}^N w(t) w^T(t) \quad R_{wy} = \sum_{t=1}^N w(t) y(t) \quad (4)$$

Assuming that $d(t)=0$, we have that $\theta_{LS}=\theta^*$ as long as the inverse of the matrix R_w exists. Otherwise, we first define

$$R_{wd} = \sum_{t=1}^N w(t) d(t) \quad (5)$$

and deduce that

$$\theta_{LS} = \theta^* + R_w^{-1} R_{wd} \quad (6)$$

Stochastic Analysis of Error: Assuming that the signal $d(t)$ is a stochastic process, the estimate θ_{LS} is itself a random variable whose properties can be predicted. Specifically, one can show [14] that if $w(t)$ is deterministic and $d(t)$ is a zero mean white noise process with variance λ^2 , one has that

$$\begin{aligned} E(\theta_{LS}) &= \theta^* \\ E((\theta_{LS} - \theta^*)(\theta_{LS} - \theta^*)^T) &= \lambda^2 R_w^{-1} \end{aligned} \quad (7)$$

The variance of the noise, λ^2 , can itself be estimated using

$$E(R_e(\theta_{LS})) = (N-n) \lambda^2 \quad (8)$$

where N is the number of data points and n is the dimension of the parameter vector. $R_e(\theta_{LS})$ is the residual error obtained for the least-squares estimate. It follows an estimate of the error in the i th component of the parameter θ is

$$(\theta_{LS} - \theta^*)_i = 2 \sqrt{\frac{(R_w^{-1})_{ii} R_e(\theta_{LS})}{N-n}} \quad (9)$$

where $(R_w^{-1})_{ii}$ is the i th diagonal element of R_w^{-1} . The factor of two assumes that 2 standard deviations is taken to be a reasonable bound for the error, although other choices are possible.

The estimate of the parametric uncertainty is easy to calculate.

The matrices R_w , R_{wy} , and $R_y = \sum_{t=1}^N y(t)^2$ can be computed recursively if needed and

$$R_e(\theta_{LS}) = R_y - \theta_{LS}^T R_w \theta_{LS} \quad (10)$$

$$= R_y - R_{wy}^T R_w^{-1} R_{wy} \quad (11)$$

The only question concerns the assumption that $w(t)$ is deterministic and $d(t)$ is a zero mean noise, which implies that $E(w(t) d(t))=0$, or $E(R_{wd})=0$. This means that the noise sequence is uncorrelated with the regressor vector $w(t)$. In practice, the signal $d(t)$ will be composed of both the measurement noise, and a term due to unmodeled effects which may be highly correlated with $w(t)$. Consider for example the process

$$y(t) = w(t) + \epsilon w^2(t) + n(t) \quad (12)$$

where $n(t)$ is a measurement noise, and $\epsilon w^2(t)$ is a small quadratic nonlinearity that perturbs the linear relationship. If we approximate this model by

$$y(t) = w(t) \theta^* + d(t) \quad (13)$$

no matter what choice of θ^* we take, the signal $d(t)$ will be composed of both the noise and a component correlated with $w(t)$ (e.g., for $\theta^*=1$, $d(t)=\epsilon w^2(t)+n(t)$).

The estimate of the parameter error will typically go to zero as time goes to infinity. Indeed, assuming that a large number of samples is taken, the matrix $(1/N) R_w$ and the scalar $(1/N) R_e$ will both tend to finite limits, so that the estimated error will tend to zero as $N \rightarrow \infty$, following $1/\sqrt{N}$. This is a correct result, assuming that the noise is uncorrelated with the regressor vector $w(t)$, so that over time, the effect of noise averages out to zero in (6). In practice, such an assumption may lead to overly optimistic estimates of the uncertainty in the parameter estimates.

Bounded Noise Analysis: An alternative approach consists in assuming that the noise $d(t)$ is bounded

$$|d(t)| \leq B \quad (14)$$

but is otherwise arbitrary. Assuming that the bound is known, each measurement leads to a constraint

$$|y(t) - w^T(t) \theta| \leq B \quad (15)$$

which can be viewed, for each t , as a subspace of the parameter space delimited by two hyperplanes. The estimate θ now becomes a *parameter set* Θ , which consists of the intersection of the subspaces obtained for all t . Keeping track of the precise parameter set is too complex and is usually replaced by an overbounding ellipsoid ([9]). The computations are more extensive than previously and the bound B on the noise must be known. Such bound may not be easy to determine, since the noise is supposed to include both the measurement noise and the contribution of the modeling uncertainty. Outliers may also be more of a problem for such algorithm, since they could lead to an empty intersection of the parameter set.

An alternative but similar approach consists in assuming an l_2 bound instead of an l_∞ bound on $d(t)$. Note that

$$\begin{aligned} R_e(\theta_{LS}) &= \sum_{i=1}^N (w^T(t) (\theta^* - \theta_{LS}) + d(t))^2 \\ &= (\theta_{LS} - \theta^*)^T R_w (\theta_{LS} - \theta^*) - 2(\theta_{LS} - \theta^*)^T R_{wd} + R_d \end{aligned} \quad (16)$$

where $R_d = \sum_{t=1}^N d(t)^2$. Using (6), it follows that

$$(\theta_{LS} - \theta^*)^T R_w (\theta_{LS} - \theta^*) = R_d - R_e(\theta_{LS}) \quad (17)$$

Now, assume that a bound $R_{d,max}$ is known such that

$$R_d = \sum_{t=1}^N d^2(t) \leq R_{d,max} \quad (18)$$

Such a bound can be viewed as an l_2 bound, or as an l_∞ bound if we take $R_{d,max} = N \cdot B^2$, with $|d(t)| \leq B$. Then,

$$(\theta_{LS} - \theta^*)^T R_w (\theta_{LS} - \theta^*) \leq R_{d,max} - R_e(\theta_{LS}) \quad (19)$$

and a simple calculation (cf. [15]) shows that (19) is satisfied if

$$(\theta_{LS} - \theta^*)_i \leq \sqrt{(R_w^{-1})_{ii} (R_{d,max} - R_e(\theta_{LS}))} \quad (20)$$

which is a new estimate of the error in the i th parameter. Note that the inequality is a "hard" bound, as opposed to the stochastic bound, although the distinction is of limited practical interest (cf. [5]). An advantage of this estimate is that it is easy to compute. On the other hand, it requires the knowledge of $R_{d,\max}$ (or B).

Sensitivity Analysis: Another view of the problem consists in looking at the *sensitivity* of residual error to the parameter θ . This approach was used to analyze data from nonlinear models of AC motors in [15], [16]. Specifically, the residual error for an arbitrary parameter θ is given by

$$R_e(\theta) = \theta^T R_w \theta - 2\theta^T R_{wy} + R_y \quad (21)$$

The minimum value of R_e is given by (10)-(11), so that (21) can be rewritten

$$R_e(\theta) = R_e(\theta_{LS}) + (\theta - \theta_{LS})^T R_w (\theta - \theta_{LS}) \quad (22)$$

We now try to determine how sensitive $R_e(\theta)$ is to variations $\theta - \theta_{LS}$. By definition, $\partial R_e / \partial \theta = 0$ around the minimum, so that we must look for another definition of sensitivity. For example, we may look at variations that lead to a doubling of the residual error (i.e., that would cause an increase in the residual error equal to what is obtained at the minimum). We have that $R_e(\theta) \leq 2 R_e(\theta_{LS})$ if and only if

$$(\theta - \theta_{LS})^T R_w (\theta - \theta_{LS}) \leq R_e(\theta_{LS}) \quad (23)$$

A sufficient condition for (23) is

$$(\theta - \theta_{LS})_i \leq \sqrt{(R_w^{-1})_{ii} R_e(\theta_{LS})} \quad (24)$$

Note that this is not an estimate of the error $\theta_{LS} - \theta^*$ but a figure of the sensitivity of the error with respect to the estimate. However, the result is remarkably similar to the previous estimates of errors. The main difference with the stochastic estimate is that the factor $(N - n)$ is not present. Its absence represents the fact that no assumption is made about the averaging, or "benevolence", of the noise. Compared to the bounded noise analysis, the sensitivity analysis does not require the *a priori* knowledge of the noise bound (although it may be argued that it implicitly assumes that $R_{d,\max} = 2 R_e(\theta_{LS})$). The factor of two in the analysis is arbitrary and other choices such as 1.1 (10% increase instead of 100%), or others, would be valid. The results would just be scaled by a constant factor.

Example: A DC Motor. Consider the example of the electrical equation describing a DC motor

$$L \frac{di}{dt} = v - R i - K_T \omega \quad (25)$$

L is the inductance (H), R is the resistance (Ω), and K_T the torque constant ($N \cdot m/A$). The signals are: v the voltage (V), i the current (A), and ω the speed (rad/s). Two experiments were performed where the voltage v was varied in 6 equal steps, from 0 to 15 V, upwards in the first case (acceleration) and downwards in the second (deceleration). The current and the angular position were measured and filtered using Butterworth filters of order 3 to reduce the pulse width modulation noise and the quantization noise. The derivatives of the current and of the position were calculated using backwards differentiation. A least-squares algorithm was used to identify L , R , and K_T , assuming (25).

The speed response $\omega(t)$ is shown on Fig. 2 and the current $i(t)$ on Fig. 3 for the acceleration experiment. The responses for the deceleration experiment are omitted for brevity.

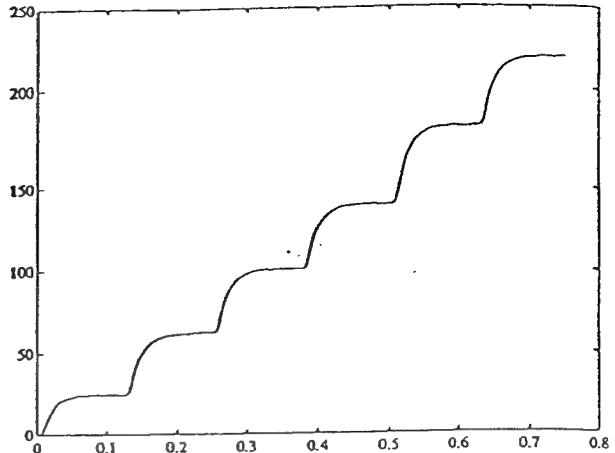


Figure 2: Speed (rad/s) as a function of time (s)

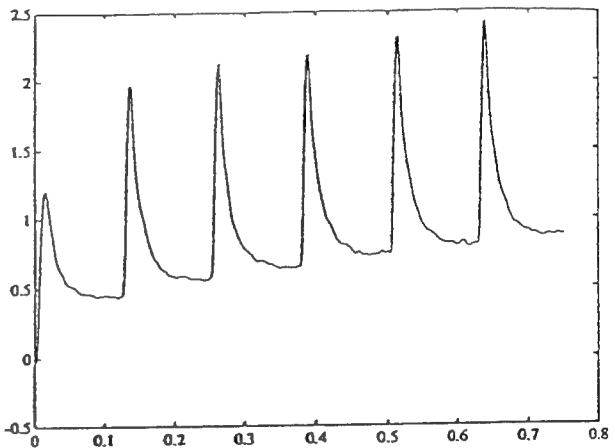


Figure 3: Current (A) as a function of time (s)

Fig. 4 shows the estimates of K_T , calculated every 100 points or 50 ms.

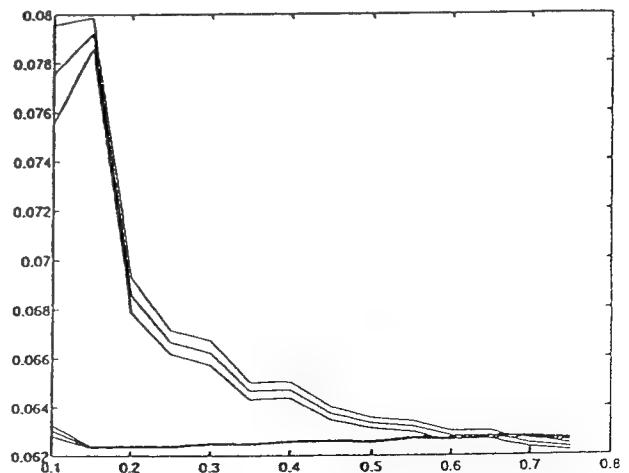


Figure 4: Torque constant ($N \cdot m/A$) estimate
Stochastic estimates of uncertainty

The 2σ bounds (9) provided by the stochastic analysis are also shown. The set of responses on the top were obtained with the

acceleration experiment, and the bottom ones with the deceleration. As one may note, the stochastic estimates are quite optimistic. Given that the signals are fairly clean (see Figs. 2 and 3), one may suspect that the residual error is largely composed of modeling error and not much of measurement noise.

Fig. 5 shows similar curves for the sensitivity-based estimates (24), which are more conservative, but more accurately reflect the uncertainty in this set of experiments.

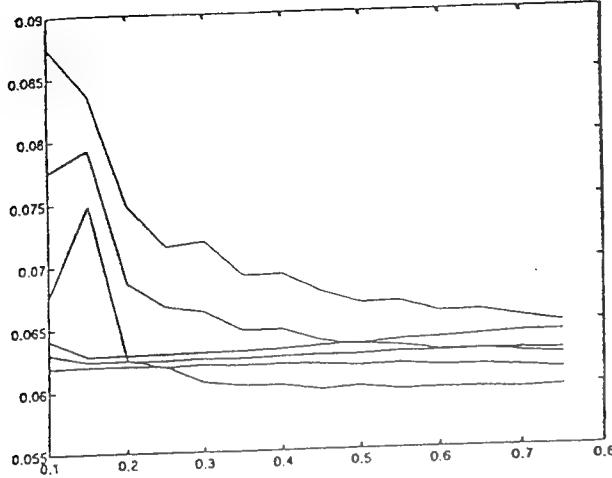


Figure 5: Torque constant (N.m/A) estimate
Sensitivity-based estimates of uncertainty

3. Reconfigurable Flight Control Systems

Reconfigurable flight control is a fascinating application of adaptive control and of methods of identification accounting for modeling uncertainty. In flight control, linearized aircraft models are typically assumed, so that one relies on a standard state-space model

$$\dot{x} = A x + B u \quad (26)$$

where the state variables are separated into longitudinal variables: α (angle of attack), q (pitch rate), h (altitude), v (velocity), and the lateral variables: β (sideslip), p (roll rate), r (yaw rate), ϕ (roll angle), ψ (yaw angle). The control inputs are also separated into the longitudinal control variable, δ_E (elevators), and the lateral control variables: δ_A (ailers) and δ_R (rudder).

After a failure, a similar linearized model is valid

$$\dot{x} = A x + B u + d \quad (27)$$

but a disturbance d is added to account for the fact that the original equilibrium is usually not an equilibrium of the failed aircraft. A new setpoint (trim) must be found. Also, the longitudinal and lateral dynamics may not be decoupled anymore, so that the A , B matrices will not be block-diagonal as for the unfailed aircraft.

Some challenges for this problem are that:

- it is a multivariable problem, with strong couplings usually appearing after failures;
- it is a nonlinear problem, which means that trim values will change after failures, but also that the linearized model will change with operating conditions, requiring the continuous use of a nonlinear or adaptive algorithm after the failure;
- the system may be highly unstable, leaving very little time for

reconfiguration;

- actuator authority is very limited and sensor noise is significant (high-gain feedback is not an option).

The use of a least-squares algorithm to identify the unknown A , B matrices and d vector is attractive and feasible. Such algorithm can very quickly lead to accurate estimates (see [17]-[19]). There is a need, however, for reliable uncertainty estimates that account for modeling errors and for methods to perform the control reconfiguration in a robust manner. We discuss some of the design trade-offs that may be considered.

Design Trade-Offs

Choice of Inputs: The elevator command δ_E is typically a symmetric command $\delta_{E,1} = \delta_{E,2} = \delta_E$. In the presence of failures, it may make sense to separate the single longitudinal command into two separate commands. The advantage is that it will then be possible to exploit all the available degrees of freedom. The disadvantage is that more parameters will need to be identified, requiring more computations and more stringent conditions for parameter convergence.

Choice of States: The so-called short period approximation (involving α and q only) and the dutch-roll approximation (involving β , p , and r only) are reduced-order models that accurately reflect aircraft dynamics over short periods of time. Their advantage is that they require less parameters and less excitation for parameter convergence. On the other hand, neglected dynamics are added to the overall dynamic uncertainty in the model.

Choice of Tracked Outputs: The choice of output variables is not unique, and a good choice may depend on speed (pitch rate at low speed, vertical acceleration at high speeds). Certain outputs lead to nonminimum phase zeros, imposing constraints on tracking performance and the choice of control algorithms.

Choice of Measured Outputs: The availability of angle of attack and sideslip measurements leads to knowledge of the full state of the system and considerable simplification of the identification and control algorithms. However, such measurements are much less reliable and more prone to damage than inertial measurements [20].

Choice of Control Objectives: Standard flight handling qualities (cf. [21]) are useful to define objectives of the reconfigured control system. However, the failed aircraft is often significantly less symmetric (and therefore decoupled) than the original one. In addition to stabilization and tracking, one must consider trim recovery and decoupling. The control objectives may also have to be adjusted according to the damage incurred to the aircraft.

Choice of Control Law: The automatic redesign of a control law is a major challenge for reconfigurable control. Making sure that such redesign will be successful in all cases and incorporating measures of uncertainty in the design are difficult problems. In addition, a tricky issue is the impact of the controller on the identification accuracy. For example, if one attempts to identify separately the effects of the two elevators $\delta_{E,1}$ and $\delta_{E,2}$, the commanded values cannot be equal or the elements of the B matrix will not be identifiable. In other words, the control law for the unfailed aircraft cannot be used. In some cases, it may also be necessary to add excitation to obtain sufficient accuracy. The best way to balance additional excitation and control performance is unfortunately an incompletely resolved issue.

Example 1: F-16 Model. We first consider simulations of a fighter aircraft using a model developed at Wright Patterson Air Force Base (see [17], [18]). The model is a linearized model of the longitudinal dynamics, with fairly detailed sensor noise models. Fig. 7 shows the estimated values of the parameter M_s . This parameter is the element of the B matrix indicating the effectiveness of the aileron in producing a pitching moment. An unfailed aircraft is assumed, and the parameters of the model are estimated using a least-squares algorithm. The solid line shows the parameter estimate using the full-order model, while the dashed line is obtained with the reduced-order model. Surprisingly, the estimate using the reduced-order model is better, despite the additional unmodeled dynamics. The condition numbers in the reduced-order case were found to be up to two orders of magnitude smaller than in the full-order case, indicating difficulties with excitation in the full-order case (ways to address this problem are discussed in [18]).

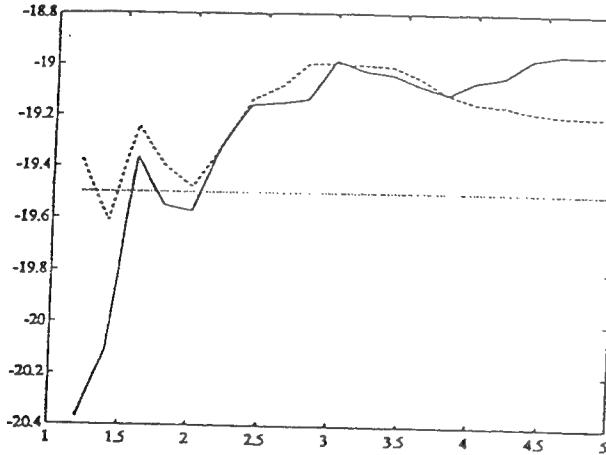


Figure 6: M_s estimates
Solid: full model; Dashed: reduced-order model

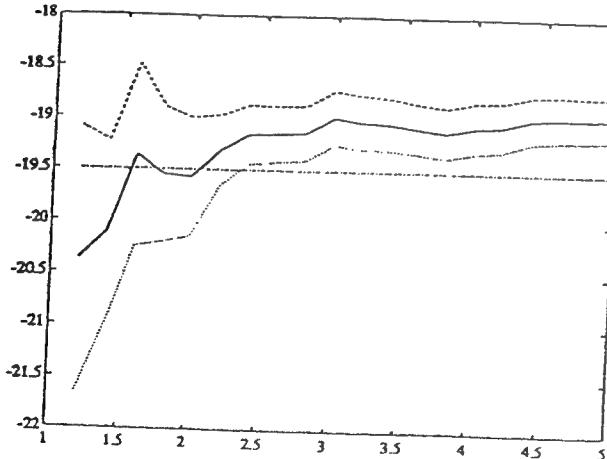


Figure 7: M_s estimate
Stochastic estimates of uncertainty

The stochastic and sensitivity estimates of uncertainty are shown in Figs. 8 and 9 respectively, for the full order model. Curves for the reduced-order model are qualitatively similar. As previously, the stochastic measures are somewhat optimistic while the sensitivity-based measures are generally conservative.

Example 2: AIAA Model. Experiments involving failures were performed using the model developed for the AIAA design

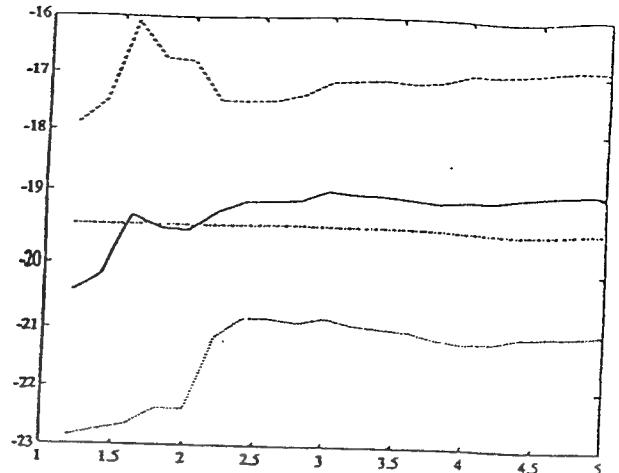


Figure 8: M_s estimate
Sensitivity-based estimates of uncertainty

challenge [22]. This is a detailed nonlinear simulation of a twin-engine fighter aircraft. We give the results here to illustrate the potential of least-squares identification methods for reconfigurable control. The reduced-order model of the unfailed aircraft was determined through a least-squares to be

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \alpha \\ q \\ \beta \\ p \\ r \end{pmatrix} &= \begin{pmatrix} -0.71 & 1.02 & 0 & 0 & 0 \\ -4.20 & -1.67 & 0 & 0 & 0 \\ 0 & 0 & 0.18 & 0.12 & -0.96 \\ 0 & 0 & -21.38 & -1.46 & 2.17 \\ 0 & 0 & 4.75 & -0.06 & -0.39 \end{pmatrix} \begin{pmatrix} \alpha \\ q \\ \beta \\ p \\ r \end{pmatrix} \\ &+ \begin{pmatrix} 0.05 & 0.00 & 0.02 \\ -6.74 & -0.06 & 0.05 \\ 0.13 & -0.12 & 0.09 \\ 0.93 & 7.87 & -0.78 \\ -0.01 & 0.10 & -2.64 \end{pmatrix} \begin{pmatrix} \delta_E \\ \delta_A \\ \delta_R \end{pmatrix} + \begin{pmatrix} 0.00 \\ 0.19 \\ 0.05 \\ -0.37 \\ -0.27 \end{pmatrix} \end{aligned}$$

The units for α , β , δ_E , δ_A , and δ_R are degrees, and for p , q and r are degrees per second. After a failure of one of the elevators, the failed aircraft was identified to be

$$\begin{aligned} \frac{d}{dt} \begin{pmatrix} \alpha \\ q \\ \beta \\ p \\ r \end{pmatrix} &= \begin{pmatrix} -0.66 & 0.99 & 0 & 0 & 0 \\ -4.48 & -1.80 & 0 & 0 & 0 \\ 0 & 0 & 0.17 & 0.11 & -0.99 \\ 0 & 0 & -22.71 & -1.66 & 1.98 \\ 0 & 0 & 4.63 & -0.08 & -0.38 \end{pmatrix} \begin{pmatrix} \alpha \\ q \\ \beta \\ p \\ r \end{pmatrix} \\ &+ \begin{pmatrix} 0.00 & 0.02 & 0.00 \\ -3.47 & -0.07 & 0.14 \\ 0.13 & -0.14 & 0.12 \\ -7.44 & 8.12 & -0.81 \\ -1.13 & 0.12 & -2.65 \end{pmatrix} \begin{pmatrix} \delta_E \\ \delta_A \\ \delta_R \end{pmatrix} + \begin{pmatrix} 0.04 \\ -1.36 \\ 0.10 \\ 3.27 \\ 0.19 \end{pmatrix} \end{aligned}$$

It was found that it was reasonable to assume that the A matrix remained block-diagonal after the failure, and this was enforced in the least-squares. On the other hand, strong couplings appeared in the B matrix, and the constant term became nonzero. The failure led to a reduction of the effectiveness of the elevators (by 50%, from -6.74 to -3.47), a rolling moment (-7.44) and a yawing moment (-1.13) appearing with the elevator command, as well as a constant pitching moment (-1.36) and a constant rolling moment (3.27). From these numbers, one can deduce that an elevator is blocked at a position of approxi

imately 1/3 (3.47/6.74) of a degree from the previous trim position. Which elevator is blocked can be deduced from the signs of the elements. However, a more interesting question for research is how the estimated parameters with the uncertainties can be used for the automatic redesign of a multivariable control law.

4. Conclusions

Obtaining reliable estimates of uncertainty in the identification of dynamic models is an important step towards building adaptive systems with good robustness and transient performance properties. In this paper, we discussed the qualitative differences between several methods to measure parametric uncertainty in the presence of model error. We discussed the advantages of a sensitivity-based analysis of the error and illustrated the results with experimental data. A challenging and exciting area of application is reconfigurable flight control. We discussed the richness of this problem through a list of the wide choices to be made, and illustrated the application of least-squares methods using two aircraft models.

5. Acknowledgements

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Multivariable Model Reference Adaptive Control with Unknown High-Frequency Gain

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Keywords: adaptive control, multivariable systems, model reference, stability.

Abstract

A multivariable model reference adaptive control algorithm is presented for the case when the high-frequency gain matrix is unknown. Only an upper bound on the norm of the matrix needs to be known *a priori*. A transformation of the parameters, with a sort of hysteresis, is used to guarantee that a controller matrix, which is normally the inverse of the estimate of the high-frequency gain matrix, remains nonsingular. It is shown that all the signals in the adaptive system are bounded and that the tracking error and the regressor error converge to zero for all bounded reference inputs. Furthermore, exponential convergence is achieved when the regressor vector is persistently exciting.

system remain bounded, that the output error converges to zero, and that the regressor error is in L_2 and converges to zero, independently of the richness of the signals used as reference inputs. We also prove that exponential convergence is achieved when persistency of excitation conditions are met. Aside from the nontrivial multivariable extensions, an original contribution of our paper is also the exponential convergence result (only asymptotic stability is achieved by Lozano *et al* (1990) algorithm).

Finally, our success in applying the hysteresis transformation also suggests that such transformation may prove helpful to solve other problems where singularity regions must be avoided, such as in adaptive pole placement, and in nonlinear control using linearization techniques.

Introduction

Single-input single-output model reference adaptive control (MRAC) results have been extended to continuous-time multivariable systems by several authors, *e.g.*, Elliot & Wolovich (1982), Singh & Narendra (1984), and Tao & Ioannou (1988). Unfortunately, current MIMO algorithms require significant *a priori* knowledge or constraints on the high-frequency gain matrix. Either the high-frequency gain matrix, K_p , must be known (fully or partially, *e.g.*, K_p diagonal with the signs of the diagonal elements known), or it must satisfy some positive definiteness condition (*e.g.*, there exists a known matrix S such that $SK_p > 0$).

In the SISO case, the problem of relaxing the requirement of knowledge of the sign of the high-frequency gain was first solved by using controllers based on the so-called Nussbaum gain, *cf.* Mudgett & Morse (1985). Because of the limited practical use of these controllers, Lozano *et al* (1990) proposed a completely different approach. In their algorithm, the controller parameters are obtained from the estimated parameters by applying a transformation with a sort of hysteresis.

In the MIMO case, the problem of unknown high-frequency gain matrix is even more difficult. The Nussbaum gain approach is not directly applicable to MIMO MRAC algorithms. However, using the hysteresis idea of Lozano *et al* (1990) with some significant modifications, we show in this paper how to design a MIMO MRAC algorithm so that stability is guaranteed even if the high-frequency gain matrix is unknown (only an upper bound on the norm of the high-frequency gain matrix and an upper bound on the norm of the matrix of unknown parameters are needed). We show that all the signals in the

1 Definitions and facts

Definition 1 : Hermite normal form

For any square, strictly proper, and nonsingular plant $P(s) \in \mathbb{R}^{p \times p}(s)$ (the set of $(p \times p)$ matrices whose elements are rational functions of s), there exists a unique matrix $H(s) \in \mathbb{R}^{p \times p}(s)$, called the *Hermite normal form*, such that:

$$P(s) = H(s)U(s)$$

$$U(s) \in \mathbb{R}^{p \times p}(s) \text{ and } \lim_{s \rightarrow \infty} U(s) = K_p \text{ nonsingular}$$

$$H(s) = \begin{bmatrix} \frac{1}{(s+a)^{r_1}} & 0 & \cdot & \cdot \\ \frac{h_{21}(s)}{(s+a)^{r_2-1}} & \frac{1}{(s+a)^{r_2}} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{(s+a)^{r_p}} \end{bmatrix}$$

where $\partial h_{ij}(s) < r_i - 1$ and a is arbitrary, but fixed *a priori*.

Definition 2 : Projections

Following Stewart & Sun (1990), Chapter I, let \mathcal{X} be a subspace of \mathbb{R}^n of dimension $r < n$ and let the columns of the orthogonal matrix $Q_{\mathcal{X}} \in \mathbb{R}^{n \times r}$ form an orthonormal basis for \mathcal{X} , with $Q_{\mathcal{X}}^T Q_{\mathcal{X}} = I$. The matrix

$$P_{\mathcal{X}} = Q_{\mathcal{X}} Q_{\mathcal{X}}^T$$

defines the *orthogonal projection onto \mathcal{X}* , *i.e.* $\forall x \in \mathbb{R}^n$, $P_{\mathcal{X}}x \in \mathcal{X}$ and $(x - P_{\mathcal{X}}x) \perp \mathcal{X}$. It can be easily verified that the matrix $P_{\mathcal{X}} \in \mathbb{R}^{n \times n}$ is symmetric, idempotent, and independent of the choice of $Q_{\mathcal{X}}$. The matrix

$$P_{\mathcal{X}}^1 = I - P_{\mathcal{X}}$$

defines the *orthogonal projection onto \mathcal{X}^1* , the subspace of \mathbb{R}^n of dimension $n - r$, that is the orthogonal complement of \mathcal{X} . The matrix $P_{\mathcal{X}}^1 \in \mathbb{R}^{n \times n}$ is also symmetric, idempotent, and independent of the specific choice of $Q_{\mathcal{X}}$.

Let $\mathcal{R}(A)$ be the *range of A* , *i.e.* the subspace of \mathbb{R}^n spanned by the columns of the matrix A . Then, there

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always exists a square orthogonal matrix U such that $AU = \begin{bmatrix} B & 0 \end{bmatrix}$ with B full rank and one has that

$$Q_{R(A)} = B(B^T B)^{-1/2} \quad P_{R(A)} = B(B^T B)^{-1} B^T$$

Definition 3 : Singular value decomposition

Let $A \in \mathbb{R}^{m \times n}$ have rank r , then the singular value decomposition of A is given by (cf. Stewart & Sun (1990), Chapter I)

$$U^T A V = \Sigma \quad U^T U = I = V^T V$$

where the matrices $U \in \mathbb{R}^{m \times \min(m,n)}$, $V \in \mathbb{R}^{n \times \min(m,n)}$, and $\Sigma = \text{diag}\{\sigma_i\}$ with

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > \sigma_{r+1} = \dots = \sigma_{\min(m,n)} = 0$$

Notation :

Assuming that $D(s) \in \mathbb{R}^{(p \times p)}[s]$ (the set of $(p \times p)$ polynomial matrices), $\partial_{\tau_i} D$ denotes the maximum polynomial degree in the i -th row of $D(s)$, $\partial_{C_i} D$ the maximum polynomial degree in the i -th column of $D(s)$, and ∂D the maximum polynomial degree in $D(s)$. In the paper, $P(s)$ denotes the transfer function of a linear time invariant operator. While $P[u]$ denotes the output of the operator (in the time domain) with input $u(t)$. Finally, if x is a vector, possibly function of time, we denote $|x(t)|$, the Euclidean norm of x at time t .

2 MRAC

2.1 Fixed controller

We use a standard controller structure (see, e.g., Elliot & Wolovich, (1982)). Assume that the plant is described by a square, nonsingular, strictly proper, and minimum phase transfer function matrix $P(s) \in \mathbb{R}^{p \times p}(s)$, assume that the Hermite normal form of $P(s)$ (see Definition 1) is known, assume that an upper bound, ν , on the maximum of the observability indices, ν_{\max} , is known. The equation of the plant is $y_p = P[u]$ and the following controller structure is considered

$$\begin{aligned} r &= M_0[r_0] \\ u &= C_0 r + \Lambda^{-1} C[u] + \Lambda^{-1} D[y_p] \end{aligned} \quad (1)$$

where $r_0 \in \mathbb{R}^p$ is a reference signal, $C_0 \in \mathbb{R}^{p \times p}$ is nonsingular, $\Lambda(s), C(s) \in \mathbb{R}^{p \times p}[s]$, $D(s) \in \mathbb{R}^{p \times p}[s]$, and $M_0(s) \in \mathbb{R}^{p \times p}(s)$. $M_0(s)$ is a proper stable transfer function matrix and $\Lambda(s)$ is a diagonal matrix of Hurwitz polynomials, $\Lambda(s) = \text{diag}\{\lambda(s)\}$, with $\partial\lambda(s) = \nu - 1$.

By combining $\{N_R, D_R\}$, a right matrix fraction description (MFD) of $P(s)$ ($P(s) = N_R(s)D_R^{-1}(s)$), with the equation of the controller (1), the output y_p can be expressed as

$$y_p = N_R((\Lambda - C)D_R - DN_R)^{-1}\Lambda C_0[r] \quad (2)$$

Now, let the reference model $M(s) = H(s)M_0(s)$, where $H(s)$ is the Hermite normal form of $P(s)$. The model output $y_m = HM_0[r_0] = H[r]$. It can be shown (see, e.g., Sastry & Bodson (1989)) that $\exists C_0^* \in \mathbb{R}^{p \times p}$, $C^*(s)$, $D^*(s) \in \mathbb{R}^{p \times p}[s]$, solution of the Diophantine equation:

$$N_R[(\Lambda - C^*)D_R - D^*N_R]^{-1}\Lambda C_0^* = H \quad (3)$$

so that model matching is achieved, with $\Lambda^{-1}D$ proper, and $\Lambda^{-1}C$ strictly proper. In particular, $C_0^* = K_P^{-1}$ nonsingular, $\partial D^* \leq \nu_{\max} - 1$, and $\partial_{\tau_i} C^* < \partial\lambda_i$. It can also be shown (cf. de Mathelin & Bodson (1992)), that the matrices C_0^*, C^*, D^* are unique if we assume knowledge of the observability indices $\{\nu_i\}$ and add the following constraint

$$\partial_{C_i} D^* \leq \nu_i - 1 \quad \forall i \quad (4)$$

2.2 Adaptive controller

In order to implement adaptation in the model reference design, we must estimate C_0^*, C^*, D^* , which satisfy the matching equality. The matching equality (3) is equivalent to

$$I = C_0^* H^{-1} P + \Lambda^{-1} C^* + \Lambda^{-1} D^* P \quad (5)$$

Define $L(s) = \text{diag}\{l(s)\}$, with $l(s)$ Hurwitz, $\partial l(s) \geq d$, where d is the maximum degree of all elements of $H^{-1}(s)$. Then multiplying both sides of (5) by L^{-1} and applying both transfer function matrices to u leads to

$$L^{-1}[u] = C_0^*(HL)^{-1}[y_p] + L^{-1}(\Lambda^{-1}C^*[u] + \Lambda^{-1}D^*[y_p]) \quad (6)$$

Since $H(s)$ is known *a priori*, (6) is an equation where the unknown parameters appear linearly which can be rewritten as

$$L^{-1}[u] = C_0^*(HL)^{-1}[y_p] + \bar{\theta}^T \bar{\psi} = \theta^T \psi \quad (7)$$

where ψ is the regressor vector and θ^* is the matrix of unknown controller parameters whose p first columns are equal to C_0^* . The following error equation can be derived

$$e_2 = \theta^T \psi - L^{-1}[u] = (\theta^T - \theta^{*T})\psi = \phi^T \psi \quad (8)$$

where θ is the estimate of θ^* and ϕ is the parameter error. We will assume that the following normalized least-squares identification algorithm with covariance resetting is used:

$$\begin{aligned} \dot{\phi} &= \dot{\theta} = -g \frac{P\psi e_2^T}{1 + \gamma\psi^T\psi} \quad \text{with } g, \gamma > 0 \\ \dot{P} &= -g \frac{P\psi\psi^T P}{1 + \gamma\psi^T\psi} \quad \text{with } P(0) = P(\tau_k) = k_0 I > 0 \end{aligned} \quad (9)$$

where e_2 is the identifier error (8) and $\{\tau_k\}$ are the resetting time instants such that

$$\{\tau_k\} = \{\tau_k \mid \lambda_{\max}(P(\tau_k^-)) = k_1 \quad \text{with } 0 \leq k_1 < k_0\}$$

where λ_{\max} denotes the largest eigenvalue. The matrix P is discontinuous at the resetting instants τ_k , with the limit on the right $P(\tau_k)$ equal to a predetermined matrix $k_0 I$. If $k_1 = 0$, the standard least-squares algorithm without resetting is obtained. For $k_1 > 0$, the algorithm will reset whenever the P matrix becomes too small.

Definition 4 : Persistent Excitation

A bounded vector ψ is persistently exciting (PE) iff $\exists \alpha > 0, \exists \delta > 0$ such that

$$\int_{\tau_0}^{\tau_0 + \delta} \psi \psi^T d\tau \geq \alpha I \quad \forall \tau_0 \geq 0$$

Definition 5 : Sufficient Excitation

A bounded vector ψ is sufficiently exciting (SE) iff $\exists \alpha > 0$ such that

$$\forall t_0 \geq 0 \ \exists \delta(t_0) > 0 \text{ such that } \int_{t_0}^{t_0+\delta} \psi \psi^T d\tau \geq \alpha I$$

Note that if a signal is PE then it is also SE, but the converse is not true.

Lemma 1 : Properties of the estimation algorithm

Assuming that $\psi \in L_{\infty}$, the estimation algorithm (9) has the following properties

1. $0 \leq P \leq k_0 I$, $k_0^{-1} I \leq P^{-1} \in L_{\infty}$, and $\frac{P\psi}{(1+\gamma\psi^T\psi)^{1/2}}, \dot{P}, \pi = \frac{P\psi}{1+\|\psi\|_{\infty}} \in L_{\infty}$.
2. $(\tau_{k+1} - \tau_k)$ is bounded below $\forall k$.
3. $\phi, \frac{\epsilon_2}{(1+\gamma\psi^T\psi)^{1/2}}, \dot{\phi}, \beta = \frac{\phi^T\psi}{1+\|\psi\|_{\infty}} \in L_{\infty}$.
4. $\frac{\epsilon_2}{(1+\gamma\psi^T\psi)^{1/2}}, \dot{\phi}, \beta \in L_2$.
5. If $\frac{\psi}{(1+\gamma\psi^T\psi)^{1/2}}$ is not SE or if $k_1 = 0$ (no resettings):
 - $\{\tau_k\}$ is a finite set.
 - $\frac{P\psi}{(1+\gamma\psi^T\psi)^{1/2}}, \dot{P}, \pi = \frac{P\psi}{1+\|\psi\|_{\infty}} \in L_2$.
 - $\dot{\phi}, \dot{P} \in L_1$.
 - $P(t)$ and $\phi(t)$ converge to some P_{∞} and ϕ_{∞} .
6. If $\frac{\psi}{(1+\gamma\psi^T\psi)^{1/2}}$ is SE:
 - $\lim_{t \rightarrow \infty} \phi(t) = 0$.
7. If $\frac{\psi}{(1+\gamma\psi^T\psi)^{1/2}}$ is PE and $k_1 > 0$:
 - $\lim_{t \rightarrow \infty} \phi(t) = 0$ exponentially.

The proof is in de Mathelin (1993).

Comments:

The specific adaptation algorithm is chosen for its special properties given in Lemma 1. Most important is the fact that the parameter error ϕ converges (although not necessarily to zero) independent of richness properties of ψ .

3 Hysteresis transformation

Although it might not be obvious *a priori*, the development of a stability proof for the MRAC algorithm reveals that the parameter C_0 in the control law (1) must be non-singular. More precisely, the matrix C_0^{-1} must remain bounded. Since this property is not guaranteed by the parameter identification algorithm (9), the parameter C_0 used in the control law will be dissociated from the estimate C_0 . Hereafter, we will use the subscript, c , to mark the controller parameters, as opposed to the estimated parameters. So, instead of C_0 being equal to the estimate C_0 , a transformation is applied to make sure that C_0 has a bounded inverse even if C_0 is close to singularity.

To complete the proof of stability and convergence of the MRAC algorithm, one finds that the following properties are required from the parameter transformation:

1. $C_0^{-1} \in L_{\infty}$.
2. $\theta_c \in L_{\infty}$.

3. $\beta_c = \frac{\phi_c^T \psi}{1+\|\psi\|_{\infty}} \in L_2 \cap L_{\infty}$, where $\phi_c = \theta_c - \theta^*$.

4. If $\lim_{t \rightarrow \infty} \phi(t) = 0$ then $\lim_{t \rightarrow \infty} \phi_c(t) = 0$.

5. If $P(t)$ and $\phi(t)$ converge to some P_{∞} and ϕ_{∞} then $\phi_c(t)$ converges to some $\phi_{c\infty}$.

Since the transformation that we propose to achieve this objective is complex, we proceed to present the transformation in several steps.

3.1 Simple hysteresis

Define $C_{0_f}(t)$ as being equal to $C_0(t)$ when $\sigma_{\min}(C_0)$ is sufficiently large, and staying constant when $\sigma_{\min}(C_0)$ becomes too small (σ_{\min} denotes the smallest singular value). More precisely, since an upper bound on $|K_p|$ is known, a lower bound, σ , on $\sigma_{\min}(C_0)$, is also known. Let $\sigma_{\min}(C_0(0)) \geq \sigma$, and define C_{0_f} such that the relationship between $\sigma_{\min}(C_{0_f})$ and $\sigma_{\min}(C_0)$ can be described by Fig. 1. Define $\{t_k\}$ as the time instants when the value of

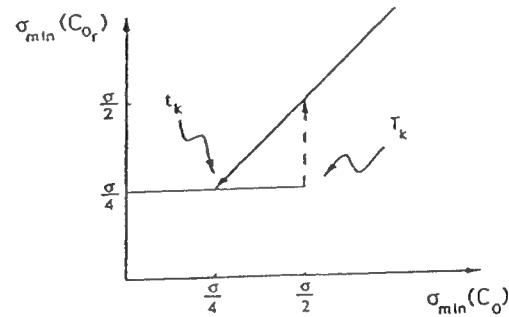


Figure 1: $\sigma_{\min}(C_{0_f})$

C_{0_f} becomes frozen and $\{T_k\}$ as the time instants when a jump in the value of C_{0_f} occurs (when C_{0_f} is unfrozen and becomes equal to C_0 again). The levels at which freezing and unfreezing occur are chosen to be different to prevent repeated switchings arbitrarily closely. Precisely stated, we let $\sigma_{\min}(C_0(0)) \geq \sigma$, $T_0 = 0$, and $\forall k \geq 1$

$$t_k \text{ is such that } \begin{cases} \sigma_{\min}(C_0(t_k)) = \sigma/b \\ \sigma_{\min}(C_0(t)) > \sigma/b \quad \forall T_{k-1} < t < t_k \end{cases}$$

$$T_k \text{ is such that } \begin{cases} \sigma_{\min}(C_0(T_k)) = \sigma/a \\ \sigma_{\min}(C_0(t)) < \sigma/a \quad \forall t_k < t < T_k \end{cases} \quad (10)$$

In Fig. 1, the parameters a and b are respectively equal to 2 and 4, but other values can be selected, provided that $1 < a < b < \infty$. Note that, since $C_0(t)$ is continuous, $t_k < T_k < t_{k+1} < T_{k+1} \quad \forall k \geq 1$ and

$$\begin{aligned} \sigma_{\min}(C_0(t)) &> \sigma/b \text{ if } T_{k-1} \leq t < t_k \\ \sigma_{\min}(C_0(t)) &< \sigma/a \text{ if } t_k \leq t < T_k \end{aligned}$$

The relationship between $C_{0_f}(t)$ and $C_0(t)$ is described by

$$C_{0_f}(t) = \begin{cases} C_0(t) & \text{if } T_{k-1} \leq t < t_k \\ C_0(t_k) & \text{if } t_k \leq t < T_k \end{cases} \quad (11)$$

so that $\sigma_{\min}(C_{0_f}(t)) \geq \sigma/b \quad \forall t$. The following parameter transformation

$$C_{0_c} = C_0, \quad \bar{\theta}_c = \bar{\theta} \quad (12)$$

would then appear to be adequate at this stage. However, it can be checked that the transformation (12) does not

guarantee property 3, although it guarantees all the others. Therefore, the following additional transformation is necessary.

3.2 Simple hysteresis + projection

To solve the problem of guaranteeing property 3, the matrix θ_c in the controller is also transformed, using the following parameter transformation

$$\theta_c(t) = \begin{cases} \theta(t) & \text{if } T_{k-1} \leq t < t_k \quad \forall k > 0 \\ \theta(t) + P(t)P_p(t)(P_p^T(t)P_p(t))^{-1}(C_{0_f}^T(t) - C_0^T(t)) \\ \quad \quad \quad \text{if } t_k \leq t < T_k \quad \forall k > 0 \end{cases} \quad (13)$$

where P_p is the rectangular matrix made of the first p columns of P . Since P is symmetric, P_p^T is the matrix made of the first p rows of P . It can be checked that, with this transformation, $C_{0_c} = C_{0_f}$. Further, the transformation is well-defined and all the necessary properties are guaranteed as long as

$$\exists \epsilon > 0 \quad \text{s.t.} \quad \sigma_{\min}(P_p^T(t)) > \epsilon \quad \forall t_k \leq t < T_k \quad (14)$$

In the SISO case, it can be proved that condition (14) is always satisfied, so that the parameter transformation (13) is always well-defined (it is the transformation presented in Lozano *et al* (1990)). Indeed, given the least-squares estimation algorithm (9), $\frac{d}{dt}(P^{-1}\phi) = 0$ between resettings, from which one can deduce that

$$C_0^T(t) = C_0^{*T} + P_p^T(t) \frac{\phi(\tau_j)}{k_0} \quad (15)$$

where τ_j is the last covariance resetting time instant, i.e. $\tau_j < t < \tau_{j+1}$. Therefore,

$$\sigma_{\min}(C_0^T(t)) \geq \sigma - \sigma_{\max}(P_p^T(t)) \frac{|\phi(0)|}{k_0} \quad (16)$$

In the SISO case, P_p is a vector. Based on (16), if for some $t > 0$, $|P_p(t)| < \sigma(1 - 1/a) \frac{k_0}{|\phi(0)|}$, then the scalar $C_0(t)$ is such that $|C_0(t)| > \sigma/a$ and $\exists k \geq 1$ such that $T_{k-1} \leq t < t_k$. Therefore, condition (14) is always satisfied.

Unfortunately, in the MIMO case, there is no guarantee that condition (14) is respected $\forall t > 0$. Indeed, if there is excitation in some directions but not others, $\sigma_{\min}(P_p(t))$ could tend to zero while $\sigma_{\max}(P_p(t))$ would not. Therefore, one could conceive that, given certain initial conditions and given certain reference signals, $\lim_{t \rightarrow \infty} \sigma_{\min}(P_p(t)) = 0$ with $\lim_{t \rightarrow \infty} \sigma_{\max}(C_0(t)) < \sigma/b$. In that case, the transformation (13) would not be well-defined $\forall t > 0$. Therefore, property 2 would not be guaranteed for this transformation in the MIMO case.

3.3 Selective hysteresis + projection

To solve the problem of the pseudo-inverse $P_p(P_p^T P_p)^{-1}$ not being necessarily bounded in the MIMO case, an additional modification is brought to the parameter transformation. Essentially, the problem is that if the estimated matrix C_0 persists in being singular, despite the presence of excitation in certain (but obviously not all) channels, then the matrix C_{0_c} must be updated selectively in the directions where there is excitation. The major challenge is to achieve this objective while maintaining the boundedness of $C_{0_c}^{-1}$ and all the necessary properties.

We consider the following transformation

$$\theta_c(t) = \begin{cases} \theta(t) & \text{if } T_{k-1} \leq t < t_k \quad \forall k > 0 \\ \theta(t) + P(t)V_P(t)\Sigma_P^{-1}(t)U_P^T(t)(C_{0_f}^T(t)R_P^T(t) - C_0^T(t)) \\ \quad \quad \quad \text{if } t_k \leq t < T_k \quad \forall k > 0 \end{cases} \quad (17)$$

where $V_P \Sigma_P^{-1} U_P^T$ is a lower rank approximation of $P_p(P_p^T P_p)^{-1}$. Specifically, we consider the following partitioned singular value decomposition of P_p^T

$$[U_P \quad U_\epsilon]^T P_p^T [V_P \quad V_\epsilon] = \begin{bmatrix} \Sigma_P & 0 \\ 0 & \Sigma_\epsilon \end{bmatrix} = \Sigma \quad (18)$$

and $\Sigma = \text{diag}\{\sigma_i\} \quad \sigma_i \geq \sigma_j \quad \forall i < j$

such that

$$\begin{aligned} \Sigma_P &= \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \sigma_{N_P} \end{bmatrix} > (\sigma_{N_P+1} + \delta_P)I \\ \Sigma_\epsilon &= \begin{bmatrix} \sigma_{N_P+1} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \sigma_{p-1} & 0 \\ 0 & \dots & 0 & \sigma_p \end{bmatrix} \\ &\leq \begin{bmatrix} \epsilon_P + (p - N_P - 1)\delta_P & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \epsilon_P + \delta_P & 0 \\ 0 & \dots & 0 & \epsilon_P \end{bmatrix} \\ &\leq (\epsilon_P + (p - N_P - 1)\delta_P)I \end{aligned} \quad (19)$$

N_P is the size of Σ_P and, therefore, $U_P \in \mathbb{R}^{p \times N_P}$, $V_P \in \mathbb{R}^{2p \times N_P}$, $U_\epsilon \in \mathbb{R}^{p \times (p - N_P)}$, and $V_\epsilon \in \mathbb{R}^{2p \times (p - N_P)}$. The constants ϵ_P and δ_P are arbitrarily chosen, but must satisfy the following conditions

$$\begin{aligned} 0 &< \epsilon_P < (1 - 1/a) \frac{\sigma_{k_0}}{(1 + (p - 1)/\alpha_P)(\sigma_{\theta^*} + |\theta(0)|)} \\ 0 &< \delta_P = \epsilon_P/\alpha_P \quad \text{with} \quad \alpha_P > 1 \end{aligned} \quad (20)$$

where σ_{θ^*} is a known upper bound of $|\theta^*|$. Basically, Σ_ϵ contains the singular values of P_p^T no greater than ϵ_P and the ones which are sufficiently close to them (by increment δ_P). Σ_P contains the larger singular values such that $\sigma_{\min}(\Sigma_P) > \sigma_{\max}(\Sigma_\epsilon) + \delta_P$. The difference δ_P is found to be required to guarantee continuity in the proof.

Finally, $R_P^T = V_f U_0^T$, where V_f is an orthonormal basis of the subspace $\mathcal{X} = \mathcal{R}(C_0, U_P U_P^T)$ of size N_P and U_0 is an orthonormal basis of the subspace $\mathcal{Y} = \mathcal{R}(C_0 U_\epsilon U_\epsilon^T)^\perp$ also of size N_P . Therefore, given the properties of projections (cf. Definition 2)

$$\begin{aligned} V_f^T V_f &= I \quad \text{and} \quad V_f V_f^T = P_{\mathcal{R}(C_0, U_P U_P^T)} \triangleq P_{\mathcal{X}} \\ U_0^T U_0 &= I \quad \text{and} \quad U_0 U_0^T = P_{\mathcal{R}(C_0 U_\epsilon U_\epsilon^T)^\perp} \triangleq P_{\mathcal{Y}} \end{aligned} \quad (21)$$

If $\sigma_{\min}(P_p^T(t)) > \epsilon_P$ then $N_P = p$, and $P_{\mathcal{X}} = P_{\mathcal{Y}} = I$. If $\sigma_{\min}(P_p^T(t)) \leq \epsilon_P$ then $N_P < p$ and $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ are given by

$$\begin{aligned} P_{\mathcal{X}} &= C_{0_f} U_P (U_P^T C_{0_f}^T C_{0_f} U_P)^{-1} U_P^T C_{0_f}^T \\ P_{\mathcal{Y}} &= (I - C_{0_f} U_P (U_P^T C_{0_f}^T C_{0_f} U_P)^{-1} U_P^T C_{0_f}^T) \end{aligned} \quad (22)$$

Indeed, referring to Definition 2, we let $A = C_{0_f} U_P U_P^T$ and $U = [U_P \quad U_\epsilon]$, so that $B = C_{0_f} U_P$ and $P_{\mathcal{X}}$ is

given by (22). This assumes that $B = C_{0_f} U_P$ has full column rank, i.e., $U_P^T C_{0_f}^T C_{0_f} U_P$ nonsingular. However, since C_{0_f} is nonsingular, $(U_P^T C_{0_f}^T C_{0_f} U_P)^{-1}$ is well-defined and the number of columns of V_f is equal to N_P . A similar derivation applies for P_y . From (15), it can be checked that

$$U_\epsilon^T C_0^T(t) = U_\epsilon^T C_0^{*T} + U_\epsilon^T P_p^T(t) \frac{\phi(r_j)}{k_0} = U_\epsilon^T C_0^{*T} + \Sigma_\epsilon V_\epsilon^T \frac{\phi(r_j)}{k_0} \quad (23)$$

where r_j is the last covariance resetting time instant, i.e. $r_j < t < r_{j+1}$. Therefore, given (19) and (20)

$$\sigma_{\min}(U_\epsilon^T C_0^T) \geq \sigma - (\epsilon_P + (p - N_P - 1)\delta_P) \frac{|\phi(0)|}{k_0} > \sigma/a \quad (24)$$

so that $(U_\epsilon^T C_0^T C_0 U_\epsilon)^{-1}$ is well-defined and the number of columns of U_0 is indeed equal to N_P .

Comments:

To guarantee some continuity properties and the uniqueness of V_f and U_0 , the Gram-Schmidt orthogonalization procedure with memory described in the appendix is applied to P_X and P_y to compute V_f and U_0 respectively. Using this procedure, we also have that if $\sigma_{\min}(P_p^T(t)) > \epsilon_P$ then $N_P = p$, $R_P^T(t) = I$, $V_P(t)\Sigma_P^{-1}(t)U_P^T(t) = P_p(t)(P_p^T(t)P_p(t))^{-1}$, and the parameter transformation (17) is simply identical to the parameter transformation (13).

It can also be shown that $N_P \geq 1$. Indeed, if $\exists t > 0$ such that $N_P = 0$, then from (24), $\sigma_{\min}(C_0) > \sigma/a$. Therefore, $\exists k > 0$ such that $T_{k-1} \leq t < t_k$ and $\theta_c(t) = \theta(t)$. In other words, if P_p was to be small in all directions, then it would mean that there had been enough excitation so that C_0 would be close to C_0^* and one would be out of the singularity region. Therefore, $N_P \geq 1$, $\sigma_{\min}(\Sigma_P) > \epsilon_P + \delta_P$ and the parameter transformation (17) is well-defined. Basically, the inverse $(P_p^T P_p)^{-1}$ has been replaced by a lower order inverse, whose existence and boundedness can be guaranteed.

Finally, it can be verified that $C_{0_c}^{-1} \in L_\infty$. Indeed, if $T_{k-1} \leq t < t_k$ or if $t_k \leq t < T_k$ and $\sigma_{\min}(P_p^T(t)) > \epsilon_P$, then $C_{0_c}^T(t) = C_{0_f}^T(t)$ whose inverse is always bounded. If $t_k \leq t < T_k$ and $\sigma_{\min}(P_p^T) \leq \epsilon_P$, then

$$\begin{aligned} C_{0_c}^T(t) &= C_0^T(t) + U_P(t)U_P^T(t)(C_{0_f}^T(t)R_P^T(t) - C_0^T(t)) \\ &= U_P(t)U_P^T(t)C_{0_f}^T(t)R_P^T(t) + U_\epsilon(t)U_\epsilon^T(t)C_0^T(t) \end{aligned} \quad (25)$$

Therefore, using (21) and (22),

$$\begin{aligned} C_{0_c}^T C_{0_c} &= U_P U_P^T C_{0_f}^T V_f V_f^T C_{0_f} U_P U_P^T + U_\epsilon U_\epsilon^T C_0^T C_0 U_\epsilon U_\epsilon^T \\ &= U \begin{bmatrix} U_P^T C_{0_f}^T C_{0_f} U_P & 0 \\ 0 & U_\epsilon^T C_0^T C_0 U_\epsilon \end{bmatrix} U^T \end{aligned} \quad (26)$$

where $U = [U_P \quad U_\epsilon]$, so that $\sigma_{\min}(C_{0_c}) \geq \sigma/b$.

Finally, given (17) and all the previous definitions, the parameter transformation with hysteresis will have the following form

$$\theta_c(t) = \theta(t) + P(t)Q(t) \quad (27)$$

$$Q(t) = \begin{cases} 0 & \text{if } T_{k-1} \leq t < t_k \quad \forall k > 0 \\ P_p(t)(P_p^T(t)P_p(t))^{-1}(C_{0_f}^T(t) - C_0^T(t)) \\ \quad \text{if } \begin{cases} t_k \leq t < T_k \quad \forall k > 0 \\ \text{and } \sigma_{\min}(P_p^T(t)) > \epsilon_P \end{cases} \\ V_P(t)\Sigma_P^{-1}(t)U_P^T(t)(C_{0_f}^T(t)R_P^T(t) - C_0^T(t)) \\ \quad \text{if } \begin{cases} t_k \leq t < T_k \quad \forall k > 0 \\ \text{and } \sigma_{\min}(P_p^T(t)) \leq \epsilon_P \end{cases} \end{cases} \quad (28)$$

Lemma 2 : Properties of the hysteresis transformation

Assuming that $\psi \in L_{\infty}$ and that the parameter estimation algorithm is defined by (9), then

1. The transformation is always uniquely defined.
2. $C_{0_c}^{-1} \in L_\infty$.
3. $Q \in L_\infty$, $\theta_c \in L_\infty$, and $\phi_c \in L_\infty$, where $\phi_c = \theta_c - \theta^*$.
4. $\beta_c = \frac{\phi_c^T \psi}{1 + \|\psi\|_\infty} \in L_2 \cap L_\infty$.
5. If $\lim_{t \rightarrow \infty} \phi(t) = 0$ then $\lim_{t \rightarrow \infty} \phi_c(t) = 0$.
6. $(t_k - T_{k-1})$ and $(T_k - t_k)$ are bounded below $\forall k$.
7. $\{T_k\}$ and $\{t_k\}$ are finite sets.
8. If $P(t)$ and $\phi(t)$ converge to some P_∞ and ϕ_∞ then $\phi_c(t)$ converges to some ϕ_{c_∞} .

The proof is in de Mathelin (1993).

Comments:

The fact that $Q \in L_\infty$ is obtained because, even though $P^{-1}(t)$ is not necessarily bounded as t increases, Σ_P^{-1} exists and is always bounded when $Q \neq 0$. In other words, should P^{-1} be unbounded, either Σ_P^{-1} will continue to exist or the algorithm will come out of the hysteresis region. The fourth property is most important: it shows that the main property on β in the estimation algorithm remains valid when the estimated parameter error ϕ is replaced by the controller parameter error ϕ_c . The fact that the intervals $(t_k - T_{k-1})$ and $(T_k - t_k)$ are bounded below comes from the normalized nature of the adaptation algorithm and from the separation of the freezing and unfreezing levels a and b . It eliminates the possibility of having an infinite number of jumps in a finite interval of time. The seventh property tells us that there is a finite number of passages in the hysteresis loop. After a while, the algorithm will settle and no more jumps will occur. The property is the consequence of the convergence of ϕ (not necessarily to zero). The algorithm could actually settle inside the hysteresis loop. In that event, we will see in the following section that the stability properties of the adaptive algorithm are preserved. The last property is obtained because of the difference δ_P between Σ_P and Σ_ϵ and because of the use of the particular Gram-Schmidt orthogonalization procedure defined in appendix.

Note that from (27) and (28) with (25),

$$\begin{aligned} U_P^T(t)C_{0_c}^T(t) &= U_P^T(t)C_{0_f}^T(t)R_P^T(t) \\ U_\epsilon^T(t)C_{0_c}^T(t) &= U_\epsilon^T(t)C_0^{*T}(t) + \Sigma_\epsilon(t)V_\epsilon^T(t) \frac{\phi(r_k)}{k_0} \end{aligned}$$

where r_k is the last covariance resetting time instant. Basically, this particular parameter transformation unfreezes C_{0_c} in the directions where $(C_0^T - C_0^{*T})$ is sufficiently small, using the knowledge that there has been sufficient excitation in those directions.

4 Stability

If we define the model signals, ψ_m , as the signals ψ when $\phi = 0$, then we can define the regressor error, e_ψ , as $e_\psi = \psi - \psi_m$.

Theorem 1 : Stability of the MRAC system

Consider the MIMO MRAC system with the parameter estimation algorithm and the hysteresis transformation described previously. If the reference input $r \in L_\infty$ and is piecewise continuous, then

- All states of the adaptive system are bounded functions of time.
- The output error $e_o = y_p - y_m \in L_\infty$ and $\lim_{t \rightarrow \infty} e_o = 0$.
- The regressor error $e_\psi = \psi - \psi_m \in L_\infty \cap L_2$ and $\lim_{t \rightarrow \infty} e_\psi = 0$.

The proof is an extension of the SISO proof of Sastry & Bodson (1989) and can be found in de Mathelin (1993).

Appendix

Gram-Schmidt orthogonalization with memory

Let $W(t) \in \mathbb{R}^{p \times p}$ be a matrix of rank r function of time and let $h > 0$ be a constant to be fixed later.

1. At time $t = 0$, apply the following procedure:

$$\begin{aligned}
 Y_1 &= W_{k(1)} \text{ with } \begin{aligned} W_{k(1)} &= k^{(1)} \text{th column of } W \text{ s.t.} \\ &|W_{k(1)}| = \max_k |W_k| \ k^{(1)} \text{ min} \end{aligned} \\
 W^{(1)} &= (I - \frac{Y_1 Y_1^T}{Y_1^T Y_1}) W \quad (W_{k(1)}^{(1)} = 0) \\
 Y_2 &= W_{k(2)}^{(1)} \text{ with } \begin{aligned} W_{k(2)}^{(1)} &= k^{(2)} \text{th column of } W^{(1)} \text{ s.t.} \\ &|W_{k(2)}^{(1)}| = \max_k |W_k^{(1)}| \ k^{(2)} \text{ min} \end{aligned} \\
 W^{(2)} &= (I - \frac{Y_2 Y_2^T}{Y_2^T Y_2}) W^{(1)} \quad (W_{k(1)}^{(2)} = W_{k(2)}^{(2)} = 0) \\
 &\vdots \\
 Y_r &= W_{k(r)}^{(r-1)} \text{ with } \begin{aligned} W_{k(r)}^{(r-1)} &= k^{(r)} \text{th col of } W^{(r-1)} \text{ s.t.} \\ &|W_{k(r)}^{(r-1)}| = \max_k |W_k^{(r-1)}| \ k^{(r)} \text{ min} \end{aligned} \\
 W^{(r)} &= (I - \frac{Y_r Y_r^T}{Y_r^T Y_r}) W^{(r-1)} = 0
 \end{aligned}$$

Then, the matrix

$$X(0) = \left[\begin{array}{cccc} \frac{Y_1}{|Y_1|} & \cdots & \frac{Y_r}{|Y_r|} \end{array} \right]$$

is an orthogonal basis of $\mathcal{R}(W(0))$, the space of dimension r spanned by the columns of $W(0)$. The order of selection of the columns of $W(0)$, $\{k^{(i)}\}_{i=0}^r$, $i = 1, \dots, r$ is uniquely defined by this procedure. Therefore, the matrix $X(0)$ is also uniquely defined.

2. Keep the initial order of selection of the columns, $\{k^{(i)}\}_{i=0}^r$, for the orthogonalization of $W(t)$, $t > 0$, until $\exists t = t_1(W) > 0 \quad \exists 1 \leq j \leq r \quad \exists 1 \leq l \leq p \text{ s.t.}$

$$|W_{k(j)}^{(j-1)}(t)|^2 + h < |W_l^{(j-1)}(t)|^2$$

where $W^{(0)} = W$. Then, at $t = t_1(W)$ the procedure defined for $t = 0$ is applied for the computation of $X(t_1)$, and

a new order of selection of the columns, $\{k^{(i)}\}_{i=t_1(W)}$, is found.

3. Finally, continue this procedure for the orthogonalization of $W(t)$, $t > t_1(W)$. The set $\{t_k(W)\}$ are the time instants when the order of selection of the columns of W is changed. The matrix $X(t)$ will be uniquely defined for all $t \geq 0$.

The procedure is applied with $W = P_X$, leading to $X = V_f$, and similarly to P_y , leading to U_0 . The advantage of using this Gram-Schmidt orthogonalization with memory is that the matrices V_f and U_0 are uniquely defined for given matrices P_X and P_y . The constant h is chosen sufficiently small that no vector Y_i may become equal to zero, while preventing the order of selection of the columns to change an infinite number of time during a finite time interval. Note that using the Gram-Schmidt orthogonalization procedure with memory for $P_X = I$ or $P_y = I$ leads to $V_f = I$ and $U_0 = I$. Finally, it can be shown (cf. de Mathelin (1993)) that if $h \leq \frac{1}{2p}$, then the Gram-Schmidt orthogonalization with memory is always well-defined for V_f and U_0 .

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Pseudo-burst Phenomenon in Ideal Adaptive Systems*

MARC BODSON†

A phenomenon is described which is similar to the well-known burst phenomenon but is observed in ideal systems, that is, adaptive systems without noise or unmodeled dynamics.

Key Words—Adaptive control; convergence analysis; harmonic analysis; Lyapunov methods; stability.

Abstract—A phenomenon is described which is similar to the well-known burst phenomenon. With examples, it is shown that the responses of adaptive systems may exhibit large transients of the error signal, separated by quiet periods. In contrast to the burst phenomenon, the responses observed in this paper correspond to ideal systems, that is, adaptive systems without noise or unmodeled dynamics. Also, in accordance with the theory, the bursts do not occur indefinitely. Two examples are discussed. The first is shown to be very similar to the burst phenomenon. The second is found to be quite different, with its origin being in the internal excitation generated through the interaction of adaptation and control. An averaging analysis is provided, which highlights the uses and limitations of averaging for the analysis of adaptive systems.

1. INTRODUCTION

THE BURST PHENOMENON in adaptive systems has been reported by several authors (see e.g. Anderson (1985) for a clear exposition in the context of adaptive identification and control and Sethares *et al.* (1989) for an interesting discussion in the context of adaptive hybrids for echo cancellation). In the presence of noise, the outputs of adaptive systems exhibit large transients at certain time intervals. The bursts are related to the parameter drift due to the noise and can be prevented using update law modifications.

It is perhaps not so well known that a very

similar phenomenon can be observed in ideal adaptive systems, that is, adaptive systems without noise or unmodeled dynamics. It is the purpose of this paper to present and explain this phenomenon, which will be called pseudo-burst phenomenon. We will discuss two examples. It will be shown that the explanation of the first example is much related to that of the burst phenomenon, although there is no parameter drift since there is no noise. We will also discuss another example, where a single burst is observed. This is a more complicated example, and we will show that its explanation lies in an unusual interaction between adaptation and control.

In Section 2, we review the burst phenomenon with a simple example. In Section 3, we present two simulations exhibiting the pseudo-burst phenomenon. In Section 4, we investigate in detail the origin of the bursts observed in the simulations and explain the pseudo-burst phenomenon. In Section 5, we investigate the application of averaging techniques. We conclude the paper with some observations about the implications of the results on the design of adaptive stabilization and control schemes.

2. THE BURST PHENOMENON

Since the explanation of the pseudo-burst phenomenon is much related to the explanation for the burst phenomenon, it is useful to review it briefly. An example will help the presentation. Both continuous-time and discrete-time examples can be found, which exhibit parameter drift and the burst phenomenon. However, simpler examples can be constructed in discrete-time, and, therefore, we present such an

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example. Consider the plant

$$x(k+1) = a^*x(k) + b^*u(k), \quad (2.1)$$

$$y(k) = x(k) \quad (2.2)$$

with the objective of tracking the output of the reference model

$$x_m(k+1) = a_m x_m(k) + b_m r(k), \quad (2.3)$$

$$y_m(k) = x_m(k) \quad (2.4)$$

where $r(k)$ is the reference input. The controller is chosen to be

$$u(k) = c(k)r(k) + d(k)y(k). \quad (2.5)$$

The closed-loop transfer function matches the reference model transfer function for the fixed value of the parameters

$$c^* = \frac{b_m}{b^*}, \quad (2.6)$$

$$d^* = \frac{(a_m - a^*)}{b^*}, \quad (2.7)$$

which are called the nominal controller parameters. A standard identification algorithm is the projection algorithm of Goodwin and Sin (1984).

$$a(k+1) = a(k) - \frac{(y(k) - a(k)y(k-1) - b(k)u(k-1))y(k-1)}{1 + u^2(k-1) + y^2(k-1)}. \quad (2.8)$$

$$b(k+1) = b(k) - \frac{(y(k) - a(k)y(k-1) - b(k)u(k-1))u(k-1)}{1 + u^2(k-1) + y^2(k-1)}, \quad (2.9)$$

where $a(k)$, $b(k)$ are the estimates of a^* and b^* at time k . The controller parameters $c(k)$, $d(k)$ are obtained from the estimates $a(k)$, $b(k)$ as if they were the true plant parameters, using (2.6)–(2.7). A modification has to be introduced to avoid division by zero if $b(k) = 0$, but this is immaterial to the present discussion. Further, it is known that $a(k) \rightarrow a^*$, $b(k) \rightarrow b^*$ and, therefore, $c(k) \rightarrow c^*$, $d(k) \rightarrow d^*$, if the plant input $u(k)$ is sufficiently rich. Here, this means that the input should have at least one sinusoidal component. It is also known that, for this scheme, the condition on the plant input $u(k)$ is equivalent to the same condition on the reference input $r(k)$.

When the input is not sufficiently rich, the parameters do not necessarily converge, even though the output error $y(k) - y_m(k)$ tends to zero as $k \rightarrow \infty$. This is not problematic, unless measurement noise is added to the adaptive

system, so that (2.2) is replaced by

$$y(k) = x(k) + n(k). \quad (2.10)$$

Then, the adaptive parameters move across the parameter space and, due to the multiplicative nature of the update laws (2.8)–(2.9), this movement often manifests itself in the form of a parameter drift. The origin of the problem is that the error

$$e(k) = y(k) - a(k)y(k-1) - b(k)u(k-1) \quad (2.11)$$

which is used for adaptation consists mostly of measurement noise, once the initial transient has passed. The parameters $a(k)$, $b(k)$ are used to calculate the controller parameters $c(k)$, $d(k)$ which, in turn, are used to control the plant. If the error (2.11) is small, the parameters move slowly and the stability of the closed-loop plant is determined by the condition that the magnitude of the closed-loop pole be less than one, i.e.

$$|a^* + b^*d(k)| = \left| a^* + \frac{b^*}{b(k)} (a_m - a(k)) \right| < 1. \quad (2.12)$$

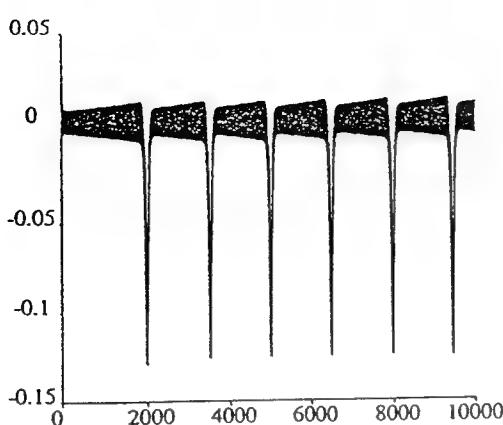
When the parameters drift, they may reach the region where (2.12) is violated, in which case the inner control loop becomes unstable and a large error burst occurs. Since the output signal becomes suddenly much larger than the noise, sufficient excitation is present and brings the parameters back to the stability region. But it is only a matter of time before the adaptive parameters drift again and reach the instability region, leading to another burst.

Figures 1 and 2 show simulations of the adaptive system with: $a^* = 0.819$, $b^* = 0.181$, $a_m = 0.670$, $b_m = 0.165$. These values correspond to the step-response equivalents of a plant $1/(s+1)$ and a reference model $1/(s+2)$ sampled at 200 msec. The noise is the sampled equivalent of a sinusoid of amplitude 0.05 and frequency 5 rad sec⁻¹ and the reference input $r(k) = 0$. Figure 1 shows the response of $x(k)$, which exhibits the large transients characteristic of the burst phenomenon. Figure 2 shows the location of the closed-loop pole

$$t_0(k) = a^* + b^*d(k). \quad (2.13)$$

It can be seen that the pole drifts across the stability border until the output becomes large enough to bring the estimates back in the stability region. The phenomenon repeats itself indefinitely.

The phenomenon of instability due to parameter drift was also reported by Rohrs *et al.*

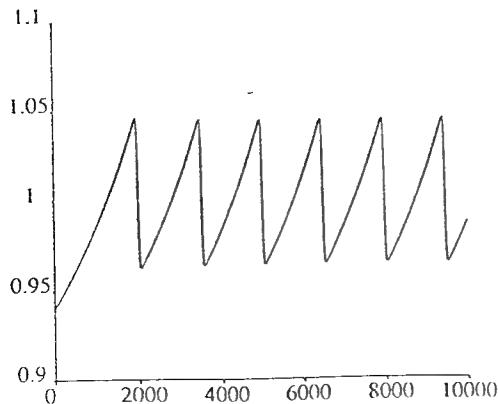
FIG. 1. Burst phenomenon, $x(k)$.

(1982) and discussed in detail by Åström (1984). The major difference between the instabilities found by Rohrs *et al.*, and the burst phenomenon described here is that, in Rohrs *et al.*, a continuous-time example is considered and it also includes unmodeled dynamics. In both cases however, the problem can be avoided by using one of the several update law modifications proposed in recent years. A deadzone stops adaptation when the error $e(k)$ is small enough to consist mostly of noise. This modification assumes that a bound on the amplitude of the noise exists and is known. More sophisticated techniques have also been proposed (see Ioannou and Sun (1988) for a review and an interesting comparison of their properties).

3. THE PSEUDO-BURST PHENOMENON

The adaptive system investigated in this section is a model reference adaptive control scheme originally proposed by Parks (1966) and extended by Narendra and Valavani (1978). We consider an n th order continuous-time system with transfer function

$$\hat{P}(s) = \frac{k_p \hat{n}_p(s)}{\hat{d}_p(s)}, \quad (3.1)$$

FIG. 2. Burst phenomenon, $t_0(k)$.

where it is assumed that $k_p > 0$, $\hat{P}(s)$ is minimum phase and the relative degree $\deg \hat{d}_p(s) - \deg \hat{n}_p(s) \triangleq n - m = 1$. The objective is to design a controller to match the output of a reference model with transfer function

$$\hat{M}(s) = \frac{k_m \hat{n}_m(s)}{\hat{d}_m(s)}, \quad (3.2)$$

where $k_m > 0$, $\hat{M}(s)$ is stable and is minimum phase, $\deg \hat{d}_m(s) = \deg \hat{d}_p(s)$ and $\deg \hat{n}_m(s) = \deg \hat{n}_p(s)$. Further, for the stability of the adaptive rule, the reference model is assumed to be strictly positive real.

We restrict ourselves to the case when the plant is of order 2 ($n = 2, m = 1$). The controller is then given by

$$u(t) = c_0(t)r(t) + c_1(t)w_1(t) + d_0(t)y_p(t) + d_1(t)w_2(t), \quad (3.3)$$

where $u(t)$ and $y_p(t)$ are the input and output of the plant and $r(t)$ is the reference input. The signals $w_1(t)$ and $w_2(t)$ are given by

$$\dot{w}_1(t) = -\lambda w_1(t) + u(t), \quad w_1(0) = 0, \quad (3.4)$$

$$\dot{w}_2(t) = -\lambda w_2(t) + y_p(t), \quad w_2(0) = 0, \quad (3.5)$$

with $\lambda > 0$ such that $\hat{n}_m(s) = s + \lambda$. The output error is given by

$$e_0(t) = y_p(t) - y_m(t), \quad (3.6)$$

where $y_m(t)$ is the output of the reference model with input $r(t)$. The adaptive parameters are updated according to

$$\dot{c}_0 = -ge_0r, \quad (3.7)$$

$$\dot{c}_1 = -ge_0w_1, \quad (3.8)$$

$$\dot{d}_0 = -ge_0y_p, \quad (3.9)$$

$$\dot{d}_1 = -ge_0w_2, \quad (3.10)$$

where $g > 0$ is called the adaptation gain.

It can be shown that there exist unique parameter values c_0^*, c_1^*, d_0^* and d_1^* such that the closed-loop transfer function matches the reference model transfer function. However, the stability of the adaptive scheme is guaranteed even if the parameters do not converge to their nominal values. In particular, it can be shown, through a Lyapunov analysis, that all state trajectories remain bounded and that $e_0(t) \rightarrow 0$ as $t \rightarrow \infty$. Further, the integral of e_0^2 from $t = 0$ to $t = \infty$ is bounded (cf. Narendra and Valavani (1978), Sastry and Bodson (1989)). A remarkable result is that the plant is not required to be stable: the scheme can be used to stabilize an arbitrary unstable system provided that the plant order is known, its gain k_p is positive, its relative degree is 1 and it is minimum phase. All states

remain bounded, whatever the system poles are and whatever the initial conditions are—although the bounds themselves depend on such parameters.

We call the system (3.1)–(3.10) the ideal adaptive system, meaning that all the assumptions on which the scheme is based are satisfied and, in particular, no noise is present. Note that the analysis results mentioned above preclude the possibility of a burst phenomenon for the ideal system, since $e_0(t) \rightarrow 0$ as $t \rightarrow \infty$. However, we will see in the following examples that, within the results of the theory, some rather unexpected phenomena can be observed.

Example 1. We first consider simulations for the adaptive system when

$$\hat{P}(s) = \frac{s+3}{(s-1)^2}, \quad (3.11)$$

$$\hat{M}(s) = \frac{s+3}{(s+1)(s+5)}. \quad (3.12)$$

The input $r = 2 \sin(t)$, the adaptation gain $g = 1$ and all initial conditions are set to zero. The parameter $\lambda = 3$ (following from (3.12) since $\hat{n}_m(s) = s + \lambda$). Figures 3 and 4 show the output error response e_0 over 250 and 1500 sec, respectively. The responses show several bursts, very reminiscent of the bursts observed in Section 2. However, the output error eventually converges to zero, consistently with the theoretical analysis which guarantees that $e_0(t) \rightarrow 0$ as $t \rightarrow \infty$ (although not at all in a monotonic manner!). Figures 5 and 6 show the responses of the adaptive parameters c_0 and d_0 , respectively. The responses of c_1 and d_1 are similar (except that the jumps at the time of the bursts are smaller[†]) and are omitted for brevity. Even-

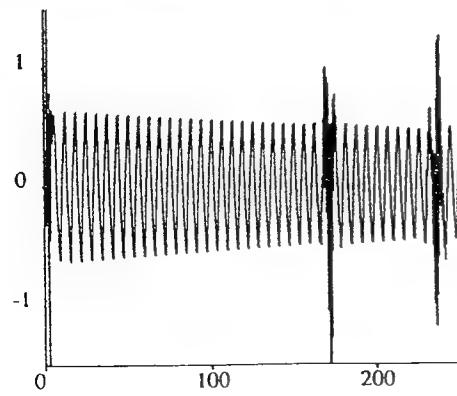


FIG. 3. Example 1, $e_0(t)$ over 250 sec.

[†] By virtue of the Lyapunov analysis, all the signals in the adaptive system are bounded, as well as their derivatives. The “jumps” observed in the responses are not discontinuities but rapid variations of signals with bounded derivatives.

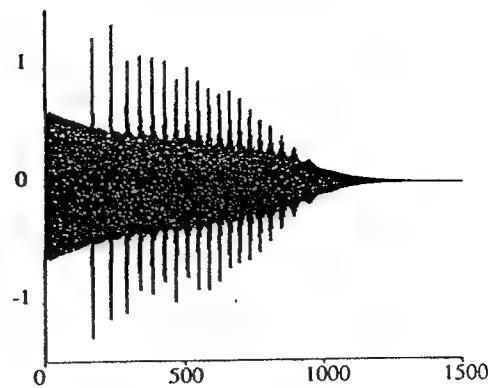


FIG. 4. Example 1, $e_0(t)$ over 1500 sec.

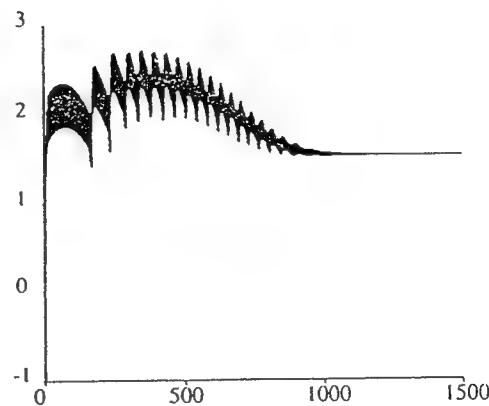


FIG. 5. Example 1, $c_0(t)$.

tually, the parameters converge to some values which are not the nominal values (those are found to be $c_0^* = 1$, $c_1^* = 0$, $d_0^* = -8$, and $d_1^* = 20$). It can be checked that the values are such that the closed-loop transfer function is stable and matches the reference model transfer function at $\omega = 1 \text{ rad sec}^{-1}$. This condition is insufficient to guarantee parameter convergence to the nominal values, but is sufficient to guarantee convergence of the output error to zero.

The bursts observed in this simulation look very similar to the bursts of Section 2 and we will see that they also have a similar explanation.

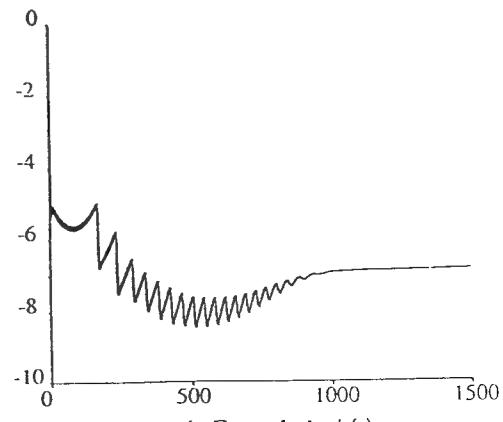
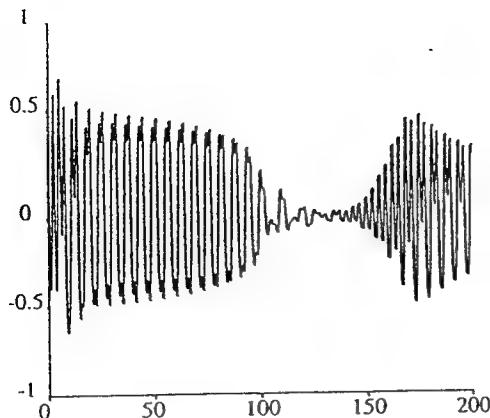
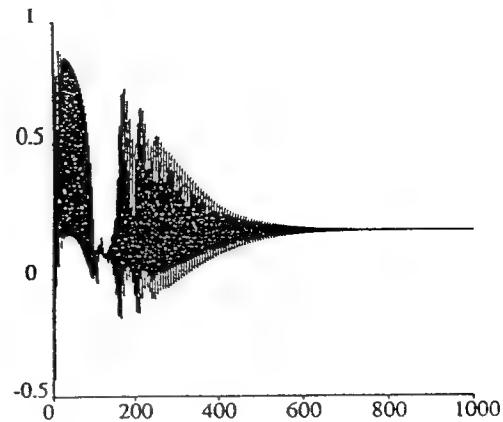


FIG. 6. Example 1, $d_0(t)$.

FIG. 7. Example 2, $e_0(t)$ over 200 sec.FIG. 9. Example 2, $c_0(t)$.

However, their origin is not in parameter drift caused by noise and the bursts cannot continue indefinitely by virtue of the analysis results. Therefore, we call this a pseudo-burst phenomenon.

Example 2. In the second simulation, the adaptive control system is identical but the plant is replaced by

$$\hat{P}(s) = \frac{s+1}{s^2+1}. \quad (3.13)$$

Figures 7 and 8 show the output error response over 200 and 1000 sec, respectively. Now the response exhibits a single burst: after 130 sec, the error reaches very small values and remains close to zero for approximately 20 sec, before taking large values again. Again, in accordance with the theoretical results, the output error eventually converges to zero and all the states are bounded.

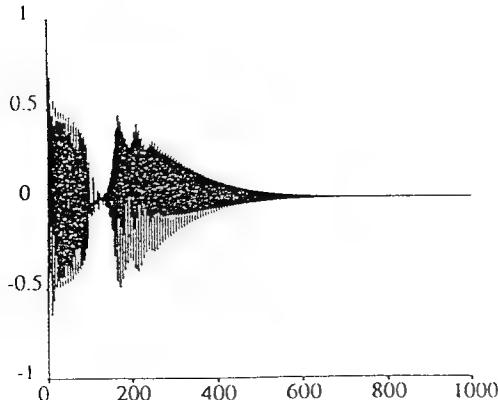
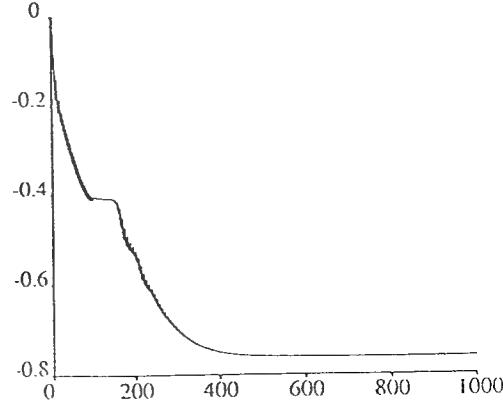
Figures 9 and 10 show the responses of the adaptive parameters c_0 and c_1 , respectively. In this case, we omitted the responses of d_0 and d_1 , which are similar to those of c_0 and c_1 (respectively). As the error reaches zero after

130 sec, the parameters seem to converge to some values before moving again. Their final values are not equal to the nominal values, equal to $c_0^* = 1$, $c_1^* = 2$, $d_0^* = -6$, and $d_1^* = 14$ in this example. The convergence of all four parameters would require at least two sinusoids. Again, it can be checked that the parameters converge to values such that the closed-loop transfer function is stable and matches the reference model transfer function at $\omega = 1 \text{ rad sec}^{-1}$. For different initial conditions, different parameter values would eventually be reached, with only a constraint to lie on a two-dimensional subspace of \mathbb{R}^4 .

The (unstable) plants of Example 1 and Example 2 are both stabilized by the same adaptive control schemes, despite completely different dynamics. However, the road to stability is not as smooth as expected. While the second example exhibits a burst similar to the bursts of the first example, we will see that its origin is quite different and raises interesting questions.

3.1. Practical relevance of the examples

As the reader may wonder whether these examples were carefully contrived, we would

FIG. 8. Example 2, $e_0(t)$ over 1000 sec.FIG. 10. Example 2, $c_1(t)$.

like to indicate how they were obtained. The examples were created to illustrate the stabilization properties of adaptive systems, as part of a homework given in a graduate course on adaptive control taught in the Fall of 1989. At the time, integer values were chosen for simplicity and no adjustments have been made since to the parameters. The fact that these examples were obtained at the first trial with unstable second order systems may be an indication that the phenomenon is quite common. In fact, after this paper was submitted for publication, Rohrs (1990) presented a discrete-time example similar to Example 1 of this paper. The paper of Hsu and Costa (1987) also presents an example of burst phenomenon for an ideal adaptive system. However, the algorithm under consideration (with σ -modification) does not guarantee Lyapunov stability of the adaptive system or convergence of the output error to zero. The phenomenon observed in the paper of Hsu and Costa is comparable to the burst phenomenon, where persistent bursting occurs because of parameter drift (in this case, the σ -modification acts like a disturbance).

The responses observed in the simulations are clearly undesirable in engineering applications. For example, adaptive stabilization algorithms have potential application to the design of self-repairing flight control systems, a subject of recent interest in the U.S.A. (cf. Chandler (1984)). Bursts and slow convergence would be highly undesirable in such an application. At issue is the transient response of the adaptive systems. Researchers are well aware of this problem, but few useful results have been obtained. Several papers have derived convergence bounds based on persistency of excitation conditions (cf. Kreisselmeier and Joos (1982); Cannetti and Espana (1989), Zang and Bitmead (1990)). These conditions are not satisfied here.

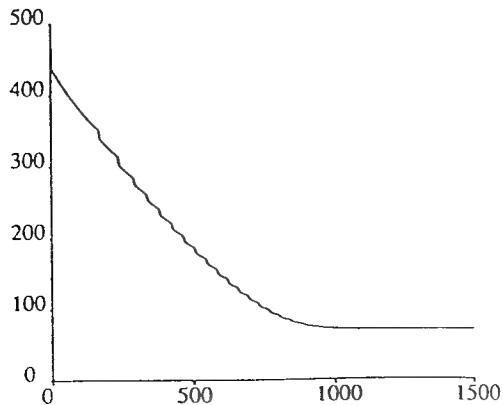


FIG. 11. Example 1, Lyapunov function $v(t)$.

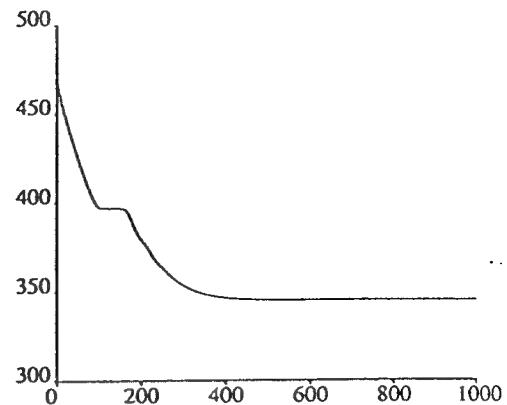


FIG. 12. Example 2, Lyapunov function $v(t)$.

Even so, the simulations do not contradict the fundamental results of the Lyapunov analysis (which is valid without the persistency of excitation condition, and is also of the source of the parameter convergence results when the condition is satisfied). To illustrate this point, Figs 11 and 12 show plots of Lyapunov functions for the systems of Examples 1 and 2, respectively. The function is indeed found to be monotonically decreasing. The simulations do not contradict the results of the available theory, but rather show the surprising responses that are possible within its scope.

3.2. Comment about the simulations

Another comment is in order about the simulations themselves. The simulations were carried out using the simulation package SIMNON. We found that the responses were quite sensitive to the error tolerance used by the integration algorithm and set by the user (cf. Elmquist *et al.* (1990)). Decreasing the value from the default of 10^{-3} , we found visible differences in the responses but they remained nearly invariant for errors less than or equal to 10^{-6} . The plots shown above were obtained for the error bound set to 10^{-8} . The integration algorithm was the Dormand-Prince 4/5 algorithm and we checked that the responses were similar to those obtained with a Runge-Kutta-Fehlberg 4/5 algorithm. It should be noted, however, that only the locations and sizes of the bursts were affected by the simulation parameters. The pseudo-burst phenomenon was observed for all values of the error tolerance that were tried, from 10^{-3} to 10^{-8} . There again, the phenomenon itself was found to be "robust" and not a singular event.

4. ANALYSIS OF THE EXAMPLES

Example 1. For the analysis of the examples, we introduce the following terminology. We call inner loop, the feedback loop consisting of the plant (3.1) with the feedback law (3.3). We will call outer loop or adaptation loop, the additional loop consisting of the update laws (3.7)–(3.10). We now proceed with the analysis of the pseudo-burst phenomenon observed in the first example. The explanation is similar to the explanation of the burst phenomenon in Section 2: the parameters evolve along trajectories which repeatedly cross the stability boundary of the inner loop. However, this occurs in the normal evolution of the adaptive parameters, and it is not caused by parameter drift due to measurement noise.

Assuming that the parameters vary slowly, the stability of the inner loop is determined by the closed-loop characteristic polynomial for frozen parameters, which is given here by

$$s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 = s^3 + (1 - c_1 - d_0)s^2 + (-5 + 2c_1 - 6d_0 - d_1)s + (3 - c_1 - 3d_1 - 9d_0). \quad (4.1)$$

Note that a third order polynomial such as (4.1) is stable if and only if $\alpha_0 > 0$, $\alpha_2 > 0$ and $t_0 > 0$, where (cf. Kamen (1990))

$$t_0 = \alpha_1 \alpha_2 - \alpha_0. \quad (4.2)$$

The variable t_0 has a similar role as the variable of the same name in Section 2. In particular, $t_0 = 0$ if and only if the polynomial has a pair of roots on the imaginary axis.

We proceed with our analysis by inspecting more closely the first burst. Figure 13 shows the detail of the output error e_0 between 150 and 200 sec, while Fig. 14 shows the evolution of t_0 . These responses are very reminiscent of Figs 1 and 2. Shortly before the burst, t_0 becomes negative, indicating that two poles of the inner loop crossed the imaginary axis. After the burst,

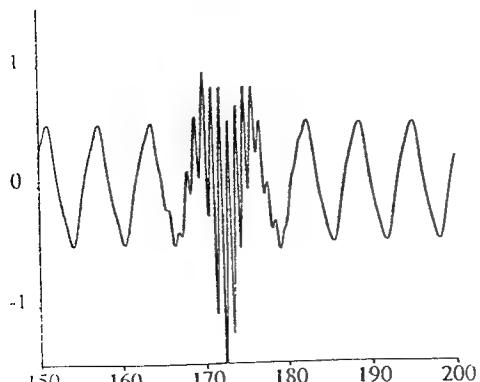


FIG. 13. Example 1, $e_0(t)$ between 150 and 200 sec.

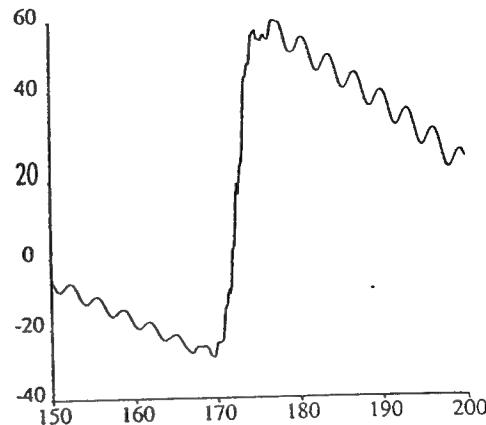


FIG. 14. Example 1, $t_0(t)$ between 150 and 200 sec.

t_0 becomes positive again, an indication that the burst has had a stabilizing effect (the other variables α_0 and α_2 were found to be positive across the burst).

A further test consists in calculating the zeros of the polynomial (4.1) for the values of the parameters immediately before and after the burst. These are given by

$$\begin{aligned} t = 168 \text{ sec (before)}: & c_0 = 1.64, \quad c_1 = 5.77, \\ & d_0 = -5.06, \quad d_1 = 2.40, \\ & s_1 = -1.01, \quad s_{2,3} = 0.36 \pm 5.92j, \\ t = 178 \text{ sec (after)}: & c_0 = 2.18, \quad c_1 = 5.31, \\ & d_0 = -6.80, \quad d_1 = 2.52, \\ & s_1 = -1.21, \quad s_{2,3} = -0.64 \pm 6.48j. \end{aligned}$$

Clearly, the burst is accompanied by the crossing of a pair of complex roots from the right to the left half plane. The detail of the burst in Fig. 13 further highlights this fact, as the period of the oscillation in the burst is approximately 1 sec which corresponds to the natural frequency of the complex roots.

In this example, the parameters repeatedly cross the stability/instability boundary. Indeed, there is no property of the algorithm that would prevent this from happening. In fact, the path followed by the parameters is known to be usually curved or spiraled from the initial conditions to the nominal parameter values (cf. the circular arcs of Åström (1984), and the averaging analysis of Sastry and Bodson (1989)).

Example 2. Given the insight gained from Example 1, we may be tempted by the following explanation: the adaptive parameters converge to values such that model matching is nearly achieved after 130 sec, but these values make the inner loop unstable and a burst results, after which the parameters converge to values which guarantee the inner loop stability. Testing inner

loop stability can be achieved as for Example 1, by testing the closed-loop characteristic polynomial, which is now given by

$$s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 = s^3 + (3 - c_1 - d_0)s^2 + (1 - 4d_0 - d_1)s + (3 - c_1 - 3d_0 - d_1). \quad (4.3)$$

However, calculation of the variables α_0 , α_2 and t_0 throughout the simulation reveals that these quantities remain positive for all time (except for a brief initial transient). In other words, the inner loop remains stable throughout the response and burst is not due to an instability of the inner loop. This means that the burst of the second example cannot be explained in the same manner as the first example.

Since the inner loop is not responsible for the instability, its origin can only be in the adaptive part of the controller or in the interactions between the inner and outer loops. The question is therefore that of the outer loop stability, or adaptation stability. A useful way to address such an issue is by means of an averaging analysis (cf. Anderson *et al.* (1986); Bodson (1988, 1989); Sastry and Bodson (1989)). Averaging allows us to approximate the time-varying differential equation describing the adaptive system by a time-invariant differential equation. The approximation of the averaged system is valid if g is sufficiently small. We will follow this approach in the next section, but let us just say for now that while averaging indeed predicts the existence of an equilibrium point (that is reached after 130 sec), it also fails to predict the burst: the burst is eliminated for small values of the gain g . However, the example has another curious property: if the adaptation gain is made larger, the burst also disappears. Figure 15 shows simulations where $g = 1$ for the first 130 sec, then $g = 0.1$ for the next 70 sec (this was done to eliminate the

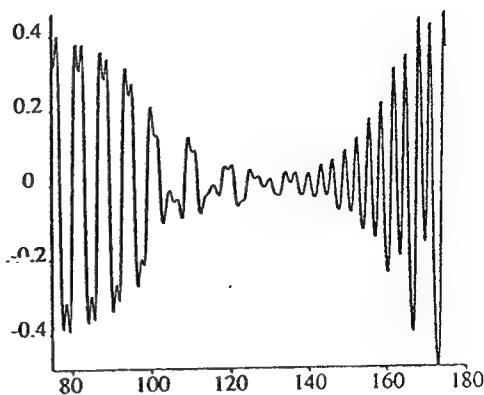


FIG. 16. Example 2, $e_0(t)$ between 75 and 175 sec.

possible effect of transients) and, for the last 100 sec, three cases are shown, that is, $g = 0.5$, 1, 2. The instability arises quickly when $g = 1$ while e_0 remains negligible in the other cases. In reality, a longer simulation reveals that a burst arises when $g = 2$, but it occurs later (after 400 sec) and is also smaller. If g is switched to 10 instead of 2, a still smaller burst occurs. In other words, there is a sort of resonance for $g = 1$.

To gain some insight into this phenomenon, we explore the responses, of the adaptive system around $t = 130$ sec. Figures 16 and 17 show e_0 and c_0 , respectively, between 75 and 175 sec. Recall that the reference input has a single frequency component at $\omega = 1 \text{ rad sec}^{-1}$. The frequency is indeed observed on e_0 before 130 sec, together with a third harmonic. On the other hand, the gain c_0 has a slowly moving component and a large second harmonic. This is easy to explain: the input u has a term $c_0 r$. Since c_0 has a second harmonic, a third harmonic is introduced at the input of the plant and is reflected at the output. The update law for c_0 is $\dot{c}_0 = -ge_0 r$, so that the correlation of the first and third harmonics of e_0 with the first harmonic

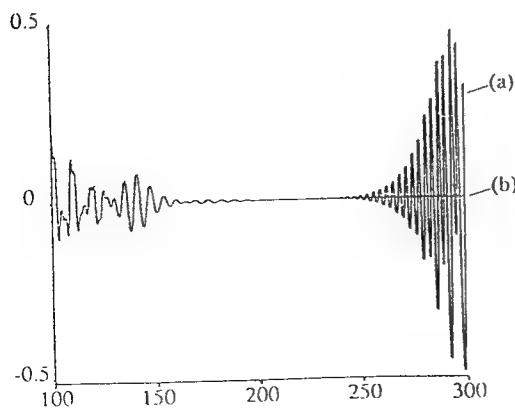


FIG. 15. Example 2, $e_0(t)$, $g = 1$ for $t = 0 \rightarrow 130$ sec, $g = 0.1$ for $t = 130 \rightarrow 200$ sec, and (a) $g = 1$ for $t = 200 \rightarrow 300$ sec, (b) $g = 0.5$ and $g = 2$ (superimposed) for $t = 200 \rightarrow 300$ sec.

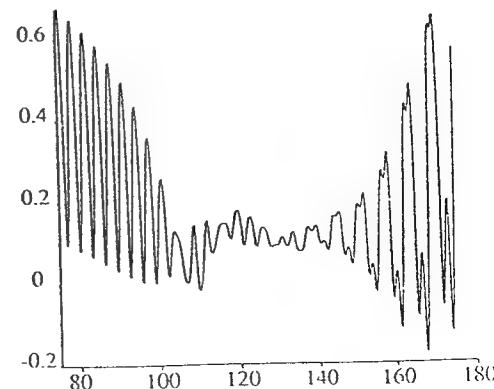


FIG. 17. Example 2, $c_0(t)$ between 75 and 175 sec.

or r creates the large second harmonic of c_0 . The puzzling fact is what happens after 130 sec. Then, for a brief while, e_0 consists of a growing oscillation of a pure second harmonic, and c_0 consists of a first and third harmonic. The situation is comparable to the situation before 130 sec, except that e_0 and c_0 have been interchanged! The output error contains a second harmonic that is not present in the reference input and that is due to the fluctuations of the adaptive parameters. We will show that it is this interaction between adaptation and control that is responsible for the unstable oscillation and for the burst.

We now provide an approximate analysis of the system, based on the observations made about the frequency content of the signals. The analysis is unconventional and exploits modulation properties of the signals. Several approximations are required for this analysis, but it should be remembered that standard methods do not provide any help, while the analysis given here leads to a reasonable prediction of the dynamic behavior observed in the adaptive system.

For $t = 130$ sec, the parameters are approximately constant, with values

$$\begin{aligned} c_0^+ &= 0.097, \quad c_1^+ \approx -0.41, \\ d_0^+ &\approx -0.62, \quad d_1^+ = 1.53. \end{aligned} \quad (4.4)$$

For these values the closed-loop transfer function is

$$\hat{P}_{cl}(s) = \frac{0.097(s+1)(s+3)}{s^3 + 4.02s^2 + 1.93s + 3.72}. \quad (4.5)$$

It can be verified that for $s = j\omega$ and $\omega = 1 \text{ rad sec}^{-1}$

$$\hat{P}_{cl}(j) = 0.32 - 0.31j = 0.35 - 0.27j = \hat{M}(j), \quad (4.6)$$

so that the transfer functions are nearly matched at the frequency of excitation. The transfer functions are not matched for all s , of course, but the closed-loop transfer function is stable, with the closed-loop poles being given by

$$s_1 = -3.77, \quad s_{2,3} = -0.13 \pm 0.99j. \quad (4.7)$$

Note that while the transfer function is stable, it has a pair of weakly damped complex poles. The explanation of the burst that we propose is that the adaptation of c_0 acts approximately like an integral feedback term (with a gain proportional to g) which destabilizes the system. It can indeed be checked, by a root-locus argument, that the system given by (4.5) with integral feedback has two unstable complex roots for some range of the feedback gain. To show that the adaptation

of c_0 acts like an integral feedback, assume that a small sinusoidal perturbation is added to the input u . Then, the corresponding perturbation on the output y_p is given by

$$\delta y_p = \frac{1}{c_0^+} \hat{P}_{cl}(s)[\delta u], \quad (4.8)$$

where $1/c_0^+ \hat{P}_{cl}$ is given by (4.4)–(4.5). This term is integrated in the adaptation through $\dot{c}_0 = -ge_0r$ and is injected back to the input through (3.3). This feedback is illustrated in Fig. 18. Note that the fluctuations of the other parameters d_0 , c_1 , and d_1 have been neglected for the sake of this argument. The system represented in Fig. 18 is not linear time-invariant. However, the feedback path is similar to a modulation/demodulation system, and this will allow us to pursue an approximate analysis in a linear time-invariant framework. Recall that $r = 2 \sin(t)$, so that for $\delta y_p = \sin(2t)$

$$\delta(e_0r) = \cos(t) - \cos(3t), \quad (4.9)$$

$$\delta c_0 = -g(\sin(t) - \frac{1}{3}\sin(3t)), \quad (4.10)$$

and

$$\delta u = g \frac{4}{3} \cos(2t) + \text{other harmonics.} \quad (4.11)$$

In other words, around $\omega = 2 \text{ rad sec}^{-1}$, the feedback can be approximated by an integral feedback $(8/3)(g/s)$. It can be checked that the transfer function $1/c_0^+ \hat{P}_{cl}(s)$ with feedback $(8/3)(g/s)$ has poles

$$s_1 = -3.68, \quad s_{2,3} = 0.14 \pm 1.86j, \quad s_4 = -0.62. \quad (4.12)$$

Therefore, the integral feedback introduced by the adaptation of c_0 destabilizes the system by moving the complex poles of (4.7) to the right half plane. The natural frequency of the oscillation is about 2 rad sec^{-1} which corresponds to the oscillation observed on Fig. 16.

This explanation offers some intriguing implications. Before and after the burst, the spectrum of the input contains frequencies not present in the reference input and introduced by

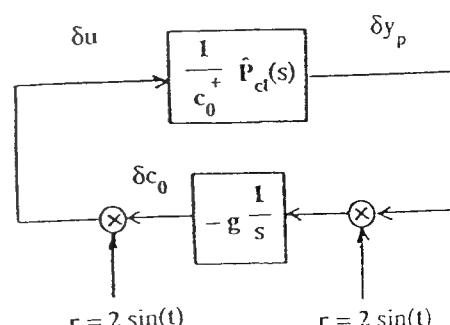


FIG. 18. Feedback created by the adaptation of c_0 .

the adaptation. The effect can be looked at from two opposite point of views. In one way, it is a destabilizing effect which prevents the adaptive system from stabilizing at the equilibrium for $t = 130$ sec. From another point of view, one may say that it provides additional internal excitation which make the parameters converge closer to their nominal values. The parameters eventually converge to values such that the closed-loop poles are given by

$$s_{1,2} = -0.22 \pm 0.98j, \quad s_3 = -4.51. \quad (4.13)$$

The closed-loop poles are better damped after the burst than before (see (4.12)) which may be considered a stabilizing effect. In either case, however, the poles remain far from the model reference poles.

5. AVERAGING ANALYSIS

Averaging has proved to be a very useful technique for the analysis of adaptive systems. Unfortunately, averaging does not provide much help in explaining the examples of this paper. On the contrary, the examples provide interesting insights into the limitations of averaging.

The averaged system corresponding to the adaptive system described in Section 2 was derived in Bodson *et al.* (1986) (also available in Sastry and Bodson (1989)). We first define the parameter error to be

$$\phi^T = (c_0 - c_0^*, c_1 - c_1^*, d_0 - d_0^*, d_1 - d_1^*). \quad (5.1)$$

The averaged system is given by (see Sastry and Bodson (1989))

$$\dot{\phi}_{av} = -g A_{av}(\phi_{av}) \phi_{av}, \quad (5.2)$$

where

$$A_{av}(\phi) = \frac{1}{2\pi c_0^*} \int_{-\infty}^{\infty} \frac{|\hat{\chi}_m(j\omega)|^2}{|\hat{\chi}_\phi(j\omega)|} (I + B(\phi)) \hat{H}_{w_{av}}^*(j\omega) \times \hat{H}_{w_{av}}^T(j\omega) (I + B^T(\phi)) \hat{M}(j\omega) S_r(d\omega), \quad (5.3)$$

where $S_r(d\omega)$ is the spectrum of the reference input. $\hat{\chi}_\phi(j\omega)$ denotes the closed-loop characteristic polynomial evaluated at $j\omega$, a function of the parameter error ϕ . $\hat{\chi}_m(j\omega)$ denotes the same polynomial for $\phi = 0$. The other matrices are given by equations (5.4) and (5.5).

$$I + B(\phi) = \begin{pmatrix} 1 & -\frac{1}{c_0^*} \phi_2 & -\frac{1}{c_0^*} \phi_3 & -\frac{1}{c_0^*} \phi_4 \\ 0 & 1 + \frac{\phi_1}{c_0^*} & 0 & 0 \\ 0 & 0 & 1 + \frac{\phi_1}{c_0^*} & 0 \\ 0 & 0 & 0 & 1 + \frac{\phi_1}{c_0^*} \end{pmatrix}, \quad (5.4)$$

$$\hat{H}_{w_{av}}(s) = \begin{pmatrix} 1 \\ \frac{1}{s + \lambda} \hat{P}^{-1}(s) \hat{M}(s) \\ \hat{M}(s) \\ \frac{1}{s + \lambda} \hat{M}(s) \end{pmatrix}. \quad (5.5)$$

In the presence of a single sinusoidal component at the input

$$r = r_0 \sin(\omega_0 t), \quad (5.6)$$

(5.2) reduces to

$$A_{av}(\phi) = \frac{r_0^2}{2c_0^*} \frac{|\hat{\chi}_m(j\omega_0)|^2}{|\hat{\chi}_\phi(j\omega_0)|^2} (I + B(\phi)) \times \text{Re} [\hat{H}_{w_{av}}^*(j\omega_0) \hat{H}_{w_{av}}^T(j\omega_0) \hat{M}(j\omega_0)] \times (I + B^T(\phi)). \quad (5.7)$$

The averaging theory is based on the assumption that the closed-loop polynomial is stable along the trajectories under consideration. However, since the expression (5.7) leads to a rational (matrix) function of ϕ , the averaged system can be defined by continuation for all ϕ except those values corresponding to poles of the closed-loop system at $s = j\omega_0$.

Figure 19 shows the responses of c_0 with c_{av} in the case of the first example. The simulations for the averaged system started at $t = 3$ sec with the same values as the parameters of the original system at this time (such that the inner loop was stabilized). It can be seen that the responses of the averaged system follow closely those of the original system but fail to reproduce the effect of the bursts. Although the averaged system can be defined by continuation across the stability boundary, we see that the assumption of a stable inner loop is necessary for the application of averaging.

Figure 20 shows the response of c_0 and c_{av} in the case of the second example and with $g = 0.1$.

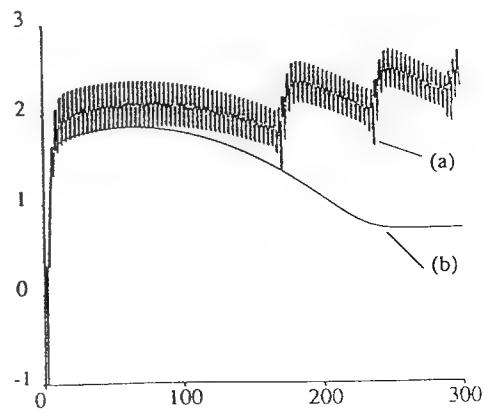
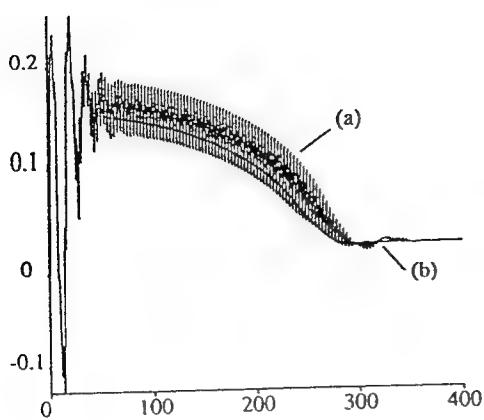


FIG. 19. Example 1, (a) $c_0(t)$ and (b) $c_{av}(t)$.

FIG. 20. Example 2, $g = 0.1$. (a) $c_u(t)$ and (b) $c_{uv}(t)$.

(the initial values for the averaged system are the same as those of the adaptive parameters after 50 sec). For this value of the adaptation gain, the original system does not exhibit a burst and we see the close approximation by the averaged system. Both systems converge to an equilibrium point. For $g = 1$, the approximation, as shown in Fig. 21, is not as good and fails to predict the burst (here the initial conditions were taken after 40 sec). The limiting theory of averaging assumes that parameters vary infinitely slowly, and therefore eliminates the internal excitation phenomena described in Section 4. While the theory only guarantees that the approximation is valid for g sufficiently small, we found plenty of situations where the predictions were accurate, even for large g (see Sastry and Bodson (1989)). This example, however, is a case where a significant difference occurs when the gain is not sufficiently small. A rule of thumb is that adaptive parameters must vary slowly compared to the reference input. If this is not the case, we see that large deviations can be caused by the interactions between the inner and outer loops.

6. CONCLUSIONS

In this paper, we discussed the existence of a phenomenon in adaptive systems that was close

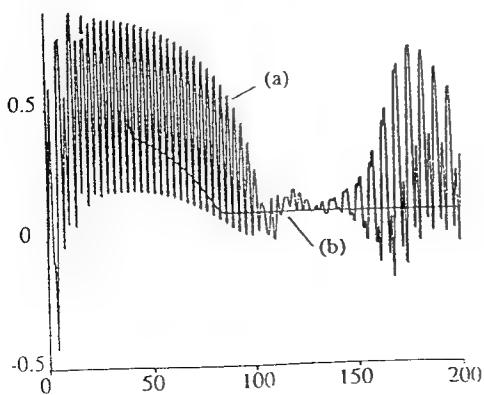
to the well-known burst phenomenon. It was also characterized by large transients of the error signal, separated by quiet periods. The main difference was that the phenomenon occurred in ideal adaptive systems, and was not caused by noise or unmodeled dynamics (or σ -modification). Therefore, we called this a pseudo-burst phenomenon. By virtue of the theoretical properties of adaptive systems, the bursts could not recur indefinitely.

The explanation of the bursts in the first example was shown to be very similar to the explanation for the burst phenomenon. When the adaptive parameters reached values which made the inner loop unstable, large transients followed and parameters eventually returned to the stable region. In this case, however, the crossing of the stability boundary was not caused by parameter drift due to noise, but occurred in the normal evolution of the parameters.

Another example was described which also bore resemblance to the burst phenomenon. Its origin was quite different however. In this case, the parameters converged to values such that the output error was nearly zero. These parameters were also such that the inner loop was stable, and such that the outer loop itself was stable for slow adaptation. For a range of adaptation gains however, fast instabilities arose, so that the parameters moved away from the equilibrium and a burst-like response of the output error followed. Eventually, the adaptive parameters converged to other values which stabilized the system. The phenomenon was explained by a peculiar interaction between adaptation and control.

The second example may be contrasted with the example of Shimkin and Feuer (1988). There, an adaptive system was considered where, in one case, the controller parameters were updated continuously and, in the other, the controller parameters were frozen over (long) periods of time and updated periodically. An effect of this procedure, called block processing, was to eliminate interactions between adaptation and control such as described above. In the case of block processing, the adaptive system was found to be stable. With continuous adaptation, the system was unstable. Although the mechanism of instability of the paper of Shimkin and Feuer was different from what we observed in this paper, it is interesting to note that the interactions between adaptation and control may sometimes eliminate the excitation available at the input of the plant, while at other times introduce additional excitation.

It is not clear whether anything can be done to prevent the pseudo-burst phenomenon of Ex-

FIG. 21. Example 2, $g = 1$. (a) $c_u(t)$ and (b) $c_{uv}(t)$.

ample 1. Since the error is not negligible between the bursts in the first example, a dead-zone would have to be so large that it would eliminate the benefits of adaptation. However, what is striking about both examples is the incredibly slow convergence of the algorithm: nearly 1000 sec are needed, while the dynamics of the inner loop are concentrated around 1 rad sec^{-1} . This is especially surprising given the amount of excitation observed in the signals. The slow convergence may possibly be attributed to the poor convergence properties of the algorithm used for adaptation. We did not explore alternate algorithms but it is possible that schemes based on least-squares algorithms would have better convergence properties (cf. Sastry and Bodson (1989)).

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Multivariable Model Reference Adaptive Control without Constraints on the High-frequency Gain Matrix

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Abstract

A multivariable model reference adaptive control algorithm is presented for the case when the high-frequency gain matrix is unknown. Only an upper bound on the norm of the matrix needs to be known *a priori*. A transformation of the parameters, with a sort of hysteresis, is used to guarantee that a controller matrix, which is normally the inverse of the estimate of the high-frequency gain matrix, remains nonsingular. It is shown that all the signals in the adaptive system are bounded and that the tracking error and the regressor error converge to zero for all bounded reference inputs. Furthermore, exponential convergence is achieved when the regressor vector is persistently exciting.

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Introduction

Single-input single-output model reference adaptive control (MRAC) results have been extended to continuous-time multivariable systems by several authors, *e.g.*, Elliot & Wolovich (1982), Singh & Narendra (1984), and Tao & Ioannou (1988). Unfortunately, current MIMO algorithms require significant *a priori* knowledge or constraints on the high-frequency gain matrix. Either the high-frequency gain matrix, K_p , must be known (fully or partially, *e.g.*, K_p diagonal with the signs of the diagonal elements known), or it must satisfy some positive definiteness condition (*e.g.*, there exists a known matrix S such that $SK_p > 0$).

In the SISO case, the problem of relaxing the requirement of knowledge of the sign of the high-frequency gain was first solved by using controllers based on the so-called Nussbaum gain, *cf.* Mudgett & Morse (1985). Because of the limited practical use of these controllers, Lozano *et al* (1990) proposed a completely different approach. In their algorithm, the controller parameters are obtained from the estimated parameters by applying a transformation with a sort of hysteresis.

In the MIMO case, the problem of unknown high-frequency gain matrix is even more difficult. The Nussbaum gain approach is not directly applicable to MIMO MRAC algorithms. However, using the hysteresis idea of Lozano *et al* (1990) with some significant modifications, we show in this paper how to design a MIMO MRAC algorithm so that stability is guaranteed even if the high-frequency gain matrix is unknown (only an upper bound on the norm of the high-frequency gain matrix and an upper bound on the norm of the matrix of unknown parameters are needed). We show that all the signals in the system remain bounded, that the output error converges to zero, and that the regressor error is in L_2 and converges to zero, independently of the richness of the signals used as reference inputs. We also prove that exponential convergence is achieved when persistency of excitation conditions are met. Aside from the nontrivial multivariable extensions, an original contribution of our paper is also the exponential convergence result (only asymptotic stability is achieved by Lozano *et al* (1990) algorithm).

Finally, our success in applying the hysteresis transformation also suggests that such transformation may prove helpful to solve other problems where singularity regions must be avoided, such as in adaptive pole placement, and in nonlinear control using linearization techniques.

1 Definitions and facts

Definition 1 : Hermite normal form

Morse (1976) showed that for any square, strictly proper, and nonsingular plant $P(s) \in \mathfrak{R}^{p \times p}(s)$ (the set of $(p \times p)$ matrices whose elements are rational functions of s), there exists a unique matrix $H(s) \in \mathfrak{R}^{p \times p}(s)$, called the *Hermite normal form*, such that:

$$\begin{aligned} P(s) &= H(s)U(s) \\ U(s) &\in \mathfrak{R}^{p \times p}(s) \text{ and } \lim_{s \rightarrow \infty} U(s) = K_p \text{ nonsingular} \end{aligned}$$

$$H(s) = \begin{bmatrix} \frac{1}{(s+a)^{r_1}} & 0 & \cdot & \cdot \\ \frac{h_{21}(s)}{(s+a)^{r_2-1}} & \frac{1}{(s+a)^{r_2}} & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \frac{1}{(s+a)^{r_p}} \end{bmatrix}$$

where $\partial h_{ij}(s) < r_i - 1$ and a is arbitrary, but fixed *a priori*.

Definition 2 : Projections

Following Stewart & Sun (1990), Chapter I, let \mathcal{X} be a subspace of \mathbb{R}^n of dimension $r < n$ and let the columns of the orthogonal matrix $Q_{\mathcal{X}} \in \mathbb{R}^{n \times r}$ form an orthonormal basis for \mathcal{X} , with $Q_{\mathcal{X}}^T Q_{\mathcal{X}} = I$. The matrix

$$P_{\mathcal{X}} = Q_{\mathcal{X}} Q_{\mathcal{X}}^T$$

defines the *orthogonal projection onto \mathcal{X}* , i.e. $\forall x \in \mathbb{R}^n$, $P_{\mathcal{X}}x \in \mathcal{X}$ and $(x - P_{\mathcal{X}}x) \perp \mathcal{X}$. It can be easily verified that the matrix $P_{\mathcal{X}} \in \mathbb{R}^{n \times n}$ is symmetric, idempotent, and independent of the choice of $Q_{\mathcal{X}}$. The matrix

$$P_{\mathcal{X}}^{\perp} = I - P_{\mathcal{X}}$$

defines the *orthogonal projection onto \mathcal{X}^{\perp}* , the subspace of \mathbb{R}^n of dimension $n - r$, that is the orthogonal complement of \mathcal{X} . The matrix $P_{\mathcal{X}}^{\perp} \in \mathbb{R}^{n \times n}$ is also symmetric, idempotent, and independent of the specific choice of $Q_{\mathcal{X}}$.

Let $\mathcal{R}(A)$ be the *range of A* , i.e. the subspace of \mathbb{R}^n spanned by the columns of the matrix A . Then, there always exists a square orthogonal matrix U such that $AU = \begin{bmatrix} B & 0 \end{bmatrix}$ with B full rank and one has that

$$Q_{\mathcal{R}(A)} = B(B^T B)^{-1/2} \quad P_{\mathcal{R}(A)} = B(B^T B)^{-1} B^T$$

Definition 3 : Singular value decomposition

Let $A \in \mathbb{R}^{m \times n}$ have rank r , then the *singular value decomposition of A* is given by (cf. Stewart & Sun (1990), Chapter I)

$$U^T A V = \Sigma \quad U^T U = I = V^T V$$

where the matrices $U \in \mathbb{R}^{m \times \min(m,n)}$, $V \in \mathbb{R}^{n \times \min(m,n)}$, and $\Sigma = \text{diag}\{\sigma_i\}$ with

$$\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_r > \sigma_{r+1} = \dots = \sigma_{\min(m,n)} = 0$$

Notation :

Assuming than $D(s) \in \mathbb{R}^{(p \times p)}[s]$ (the set of $(p \times p)$ polynomial matrices), $\partial r_i D$ denotes the maximum polynomial degree in the i -th row of $D(s)$, $\partial c_i D$ the maximum polynomial degree in the i -th column of $D(s)$, and ∂D the maximum polynomial degree in $D(s)$. In the paper, $P(s)$ denotes the transfer function of a linear time invariant operator. While $P[u]$ denotes the output of the operator (in the time domain) with input $u(t)$. Finally, if x is a vector, possibly function of time, we denote $|x(t)|$, the Euclidean norm of x at time t .

2 Model reference adaptive control

2.1 Fixed controller

We use a standard controller structure (see, e.g., Elliot & Wolovich, (1982)). Assume that the plant is described by a square, nonsingular, strictly proper, and minimum phase transfer function matrix $P(s) \in \mathbb{R}^{p \times p}(s)$, assume that the Hermite normal form of $P(s)$ (see Definition 1) is known, assume that an upper bound, ν , on the maximum of the observability indices, ν_{\max} , is known. The equation of the plant is $y_p = P[u]$ and the following controller structure is considered

$$\begin{aligned} r &= M_0[r_0] \\ u &= C_0r + \Lambda^{-1}C[u] + \Lambda^{-1}D[y_p] \end{aligned} \quad (1)$$

where $r_0 \in \mathbb{R}^p$ is a reference signal, $C_0 \in \mathbb{R}^{p \times p}$ is nonsingular, $\Lambda(s), C(s) \in \mathbb{R}^{p \times p}[s]$, $D(s) \in \mathbb{R}^{p \times p}[s]$, and $M_0(s) \in \mathbb{R}^{p \times p}(s)$. $M_0(s)$ is a proper stable transfer function matrix and $\Lambda(s)$ is a diagonal matrix of Hurwitz polynomials, $\Lambda(s) = \text{diag}\{\lambda(s)\}$, with $\partial\lambda(s) = \nu - 1$.

By combining $\{N_R, D_R\}$, a right matrix fraction description (MFD) of $P(s)$ ($P(s) = N_R(s)D_R^{-1}(s)$), with the equation of the controller (1), the output y_p can be expressed as

$$y_p = N_R((\Lambda - C)D_R - DN_R)^{-1}\Lambda C_0[r] \quad (2)$$

Now, let the reference model $M(s) = H(s)M_0(s)$, where $H(s)$ is the Hermite normal form of $P(s)$. The model output $y_m = HM_0[r_0] = H[r]$. It can be shown (see, e.g, Sastry & Bodson (1989)) that $\exists C_0^* \in \mathbb{R}^{p \times p}$, $C^*(s), D^*(s) \in \mathbb{R}^{p \times p}[s]$, solution of the Diophantine equation:

$$N_R[(\Lambda - C^*)D_R - D^*N_R]^{-1}\Lambda C_0^* = H \quad (3)$$

so that model matching is achieved, with $\Lambda^{-1}D$ proper, and $\Lambda^{-1}C$ strictly proper. In particular, $C_0^* = K_p^{-1}$ nonsingular, $\partial D^* \leq \nu_{\max} - 1$, and $\partial r_i C^* < \partial \lambda_i$. It can also be shown (cf. de Mathelin & Bodson (1992)), that the matrices C_0^*, C^*, D^* are *unique* if we assume knowledge of the observability indices $\{\nu_i\}$ and add the following constraint

$$\partial c_i D^* \leq \nu_i - 1 \quad \forall i \quad (4)$$

2.2 Adaptive controller

In order to implement adaptation in the model reference design, we must estimate C_0^*, C^*, D^* , which satisfy the matching equality. The matching equality (3) is equivalent to

$$I = C_0^*H^{-1}P + \Lambda^{-1}C^* + \Lambda^{-1}D^*P \quad (5)$$

Define $L(s) = \text{diag}\{l(s)\}$, with $l(s)$ Hurwitz, $\partial l(s) \geq d$, where d is the maximum degree of all elements of $H^{-1}(s)$. Then multiplying both sides of (5) by L^{-1} and applying both transfer function matrices to u leads to

$$L^{-1}[u] = C_0^*(HL)^{-1}[y_p] + L^{-1}(\Lambda^{-1}C^*[u] + \Lambda^{-1}D^*[y_p]) \quad (6)$$

Since $H(s)$ is known *a priori*, (6) is an equation where the unknown parameters appear linearly which can be rewritten as

$$L^{-1}[u] = C_0^*(HL)^{-1}[y_p] + \bar{\theta}^{*T} \bar{\psi} = \theta^{*T} \psi \quad (7)$$

where ψ is the regressor vector and θ^* is the matrix of unknown controller parameters whose p first columns are equal to C_0^* . The following error equation can be derived

$$e_2 = \theta^T \psi - L^{-1}[u] = (\theta^T - \theta^{*T}) \psi = \phi^T \psi \quad (8)$$

where θ is the estimate of θ^* and ϕ is the parameter error. At this point, θ can be computed using a standard estimation algorithm, such as the least-squares described in the next section.

2.3 Parameter identification algorithm

We will assume that the following normalized least-squares identification algorithm with covariance resetting is used:

$$\begin{aligned} \dot{\phi} &= \dot{\theta} = -g \frac{P \psi e_2^T}{1 + \gamma \psi^T \psi} \quad \text{with } g, \gamma > 0 \\ \dot{P} &= -g \frac{P \psi \psi^T P}{1 + \gamma \psi^T \psi} \quad \text{with } P(0) = P(\tau_k) = k_0 I > 0 \end{aligned} \quad (9)$$

where e_2 is the identifier error (8) and $\{\tau_k\}$ are the resetting time instants such that

$$\{\tau_k\} = \{\tau_k \mid \lambda_{\max}(P(\tau_k^-)) = k_1 \quad \text{with } 0 \leq k_1 < k_0\}$$

where λ_{\max} denotes the largest eigenvalue. The matrix P is discontinuous at the resetting instants τ_k , with the limit on the right $P(\tau_k)$ equal to a predetermined matrix $k_0 I$. If $k_1 = 0$, the standard least-squares algorithm without resetting is obtained. For $k_1 > 0$, the algorithm will reset whenever the P matrix becomes too small.

Definition 4 : Persistent Excitation

A bounded vector ψ is *persistent exciting* (PE) iff

$$\exists \alpha > 0 \quad \exists \delta > 0 \quad \text{such that} \quad \int_{t_0}^{t_0 + \delta} \psi \psi^T d\tau \geq \alpha I \quad \forall t_0 \geq 0$$

Definition 5 : Sufficient Excitation

A bounded vector ψ is *sufficiently exciting* (SE) iff

$$\exists \alpha > 0 \quad \text{such that} \quad \forall t_0 \geq 0 \quad \exists \delta(t_0) > 0 \quad \text{such that} \quad \int_{t_0}^{t_0 + \delta} \psi \psi^T d\tau \geq \alpha I$$

Note that if a signal is PE then it is also SE, but the converse is not true.

Lemma 1 : Properties of the estimation algorithm

Assuming that $\psi \in L_{\infty}$, the estimation algorithm (9) has the following properties

1. $0 \leq P \leq k_0 I$, $k_0^{-1} I \leq P^{-1} \in L_{\infty c}$, and $\frac{P\psi}{(1+\gamma\psi^T\psi)^{1/2}}, \dot{P}, \pi = \frac{P\psi}{1+\|\psi\|_\infty} \in L_\infty$.
2. $(\tau_{k+1} - \tau_k)$ is bounded below $\forall k$.
3. $\phi, \frac{e_2}{(1+\gamma\psi^T\psi)^{1/2}}, \dot{\phi}, \beta = \frac{\phi^T\psi}{1+\|\psi\|_\infty} \in L_\infty$.
4. $\frac{e_2}{(1+\gamma\psi^T\psi)^{1/2}}, \dot{\phi}, \beta \in L_2$.
5. If $\frac{\psi}{(1+\gamma\psi^T\psi)^{1/2}}$ is not SE or if $k_1 = 0$ (no resettings):
 - $\{\tau_k\}$ is a finite set.
 - $\frac{P\psi}{(1+\gamma\psi^T\psi)^{1/2}}, \dot{P}, \pi = \frac{P\psi}{1+\|\psi\|_\infty} \in L_2$.
 - $\dot{\phi}, \dot{P} \in L_1$.
 - $P(t)$ and $\phi(t)$ converge to some P_∞ and ϕ_∞ .
6. If $\frac{\psi}{(1+\gamma\psi^T\psi)^{1/2}}$ is SE:
 - $\lim_{t \rightarrow \infty} \phi(t) = 0$.
7. If $\frac{\psi}{(1+\gamma\psi^T\psi)^{1/2}}$ is PE and $k_1 > 0$:
 - $\lim_{t \rightarrow \infty} \phi(t) = 0$ exponentially.

The proof is in de Mathelin (1993).

Comments:

The specific adaptation algorithm is chosen for its special properties given in Lemma 1. Most important is the fact that the parameter error ϕ converges (although not necessarily to zero) independent of richness properties of ψ . It may converge to zero or to some other value. Also, if ψ is not SE or if $k_1 = 0$ (no resettings) $P\psi$ will have the same properties as e_2 and there will be a finite number of resettings. These properties result from the use of $\lambda_{\max}(P(t))$ for the resetting instead of $\lambda_{\min}(P(t))$ as in Sastry & Bodson (1989). When there is not sufficient excitation, the algorithm will not reset the covariance after some time and the algorithm has the same properties as when covariance resetting is not used. Finally, it should be noted that the exponential convergence under PE conditions is guaranteed (rather than asymptotic convergence) when the covariance resetting is used. However, the stability results will be valid whether the resetting is used or not and whether there is excitation or not.

3 Hysteresis transformation

Although it might not be obvious *a priori*, the development of a stability proof for the MRAC algorithm reveals that the parameter C_0 in the control law (1) must be nonsingular. More precisely, the matrix C_0^{-1} must remain bounded. Since this property is not guaranteed by the parameter identification algorithm (9), the parameter C_0 used in the control law will be dissociated from the estimate C_0 . Hereafter, we will use the subscript, c , to mark the

controller parameters, as opposed to the estimated parameters. So, instead of C_{0c} being equal to the estimate C_0 , a transformation is applied to make sure that C_{0c} has a bounded inverse even if C_0 is close to singularity.

3.1 Necessary properties

To complete the proof of stability and convergence of the MRAC algorithm, one finds that the following properties are required from the parameter transformation:

1. $C_{0c}^{-1} \in L_\infty$.
2. $\theta_c \in L_\infty$.
3. $\beta_c = \frac{\phi_c^T \psi}{1 + \|\psi\|_\infty} \in L_2 \cap L_\infty$, where $\phi_c = \theta_c - \theta^*$.
4. If $\lim_{t \rightarrow \infty} \phi(t) = 0$ then $\lim_{t \rightarrow \infty} \phi_c(t) = 0$.
5. If $P(t)$ and $\phi(t)$ converge to some P_∞ and ϕ_∞ then $\phi_c(t)$ converges to some $\phi_{c\infty}$.

Since the transformation that we propose to achieve this objective is complex, we proceed to present the transformation in several steps.

3.2 Simple hysteresis

Define $C_{0f}(t)$ as being equal to $C_0(t)$ when $\sigma_{\min}(C_0)$ is sufficiently large, and staying constant when $\sigma_{\min}(C_0)$ becomes too small (σ_{\min} denotes the smallest singular value). More precisely, since an upper bound on $|K_p|$ is known, a lower bound, σ , on $\sigma_{\min}(C_0^*)$, is also known. Let $\sigma_{\min}(C_0(0)) \geq \sigma$, and define C_{0f} such that the relationship between $\sigma_{\min}(C_{0f})$ and $\sigma_{\min}(C_0)$ can be described by Fig. 1. Define $\{t_k\}$ as the time instants when the value of C_{0f} becomes frozen and $\{T_k\}$ as the time instants when a jump in the value of C_{0f} occurs (when C_{0f} is unfrozen and becomes equal to C_0 again). The levels at which freezing and unfreezing occur are chosen to be different to prevent repeated switchings arbitrarily closely. Precisely stated, we let $\sigma_{\min}(C_0(0)) \geq \sigma$, $T_0 = 0$, and $\forall k \geq 1$

$$\begin{aligned} t_k \text{ is such that } & \begin{cases} \sigma_{\min}(C_0(t_k)) = \sigma/b \\ \sigma_{\min}(C_0(t)) > \sigma/b \quad \forall T_{k-1} < t < t_k \end{cases} \\ T_k \text{ is such that } & \begin{cases} \sigma_{\min}(C_0(T_k)) = \sigma/a \\ \sigma_{\min}(C_0(t)) < \sigma/a \quad \forall t_k < t < T_k \end{cases} \end{aligned} \quad (10)$$

In Fig. 1, the parameters a and b are respectively equal to 2 and 4, but other values can be selected, provided that $1 < a < b < \infty$. Note that, since $C_0(t)$ is continuous, $t_k < T_k < t_{k+1} < T_{k+1} \forall k \geq 1$ and

$$\begin{aligned} \sigma_{\min}(C_0(t)) & > \sigma/b \quad \text{if } T_{k-1} \leq t < t_k \\ \sigma_{\min}(C_0(t)) & < \sigma/a \quad \text{if } t_k \leq t < T_k \end{aligned}$$

The relationship between $C_{0f}(t)$ and $C_0(t)$ is described by

$$C_{0f}(t) = \begin{cases} C_0(t) & \text{if } T_{k-1} \leq t < t_k \\ C_0(t_k) & \text{if } t_k \leq t < T_k \end{cases} \quad (11)$$

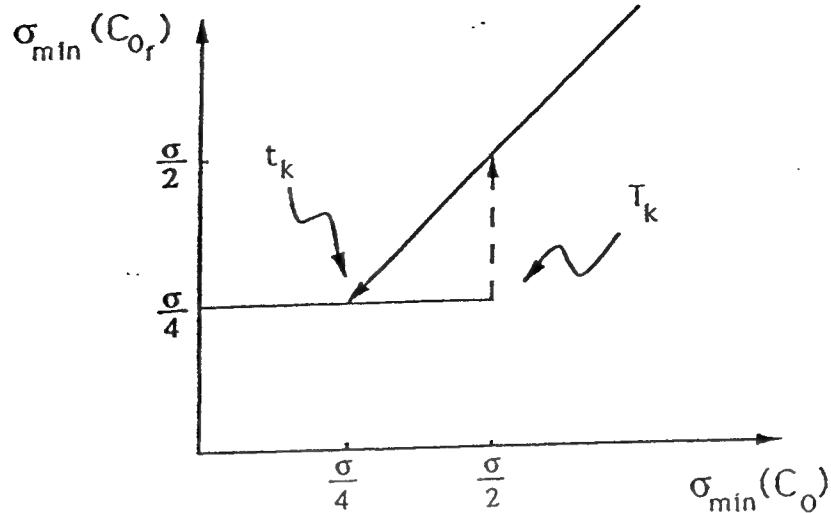


Figure 1: $\sigma_{\min}(C_{0_f})$

so that $\sigma_{\min}(C_{0_f}(t)) \geq \sigma/b \forall t$. The following parameter transformation

$$C_{0_c} = C_{0_f} \quad \bar{\theta}_c = \bar{\theta} \quad (12)$$

would then appear to be adequate at this stage. However, it can be checked that the transformation (12) does not guarantee property 3 of section 3.1, although it guarantees all the others. Therefore, the following additional transformation is necessary.

3.3 Simple hysteresis + projection

To solve the problem of guaranteeing property 3, the matrix $\bar{\theta}_c$ in the controller is also transformed, using the following parameter transformation

$$\theta_c(t) = \begin{cases} \theta(t) & \text{if } T_{k-1} \leq t < t_k \quad \forall k > 0 \\ \theta(t) + P(t)P_p(t)(P_p^T(t)P_p(t))^{-1}(C_{0_f}^T(t) - C_0^T(t)) & \text{if } t_k \leq t < T_k \quad \forall k > 0 \end{cases} \quad (13)$$

where P_p is the rectangular matrix made of the first p columns of P . Since P is symmetric, P_p^T is the matrix made of the first p rows of P . It can be checked that, with this transformation, $C_{0_c} = C_{0_f}$. Further, the transformation is well-defined and all the necessary properties of section 3.1 are guaranteed as long as

$$\exists \epsilon > 0 \quad \text{s.t.} \quad \sigma_{\min}(P_p^T(t)) > \epsilon \quad \forall t_k \leq t < T_k \quad (14)$$

In the SISO case, it can be proved that condition (14) is always satisfied, so that the parameter transformation (13) is always well-defined (it is the transformation presented in Lozano *et al* (1990)). Indeed, given the least-squares estimation algorithm (9), $\frac{d}{dt}(P^{-1}\phi) = 0$ between resettings, from which one can deduce that

$$C_0^T(t) = C_0^{*T} + P_p^T(t) \frac{\phi(\tau_j)}{k_0} \quad (15)$$

where τ_j is the last covariance resetting time instant, i.e. $\tau_j < t < \tau_{j+1}$. Therefore,

$$\sigma_{\min}(C_0^T(t)) \geq \sigma - \sigma_{\max}(P_p^T(t)) \frac{|\phi(0)|}{k_0} \quad (16)$$

In the SISO case, P_p is a vector. Based on (16), if for some $t > 0$, $|P_p(t)| < \sigma(1 - 1/a) \frac{k_0}{|\phi(0)|}$, then the scalar $C_0(t)$ is such that $|C_0(t)| > \sigma/a$ and $\exists k \geq 1$ such that $T_{k-1} \leq t < t_k$. Therefore, condition (14) is always satisfied.

Unfortunately, in the MIMO case, there is no guarantee that condition (14) is respected $\forall t > 0$. Indeed, if there is excitation in some directions but not others, $\sigma_{\min}(P_p(t))$ could tend to zero while $\sigma_{\max}(P_p(t))$ would not. Therefore, one could conceive that, given certain initial conditions and given certain reference signals, $\lim_{t \rightarrow \infty} \sigma_{\min}(P_p(t)) = 0$ with $\lim_{t \rightarrow \infty} \sigma_{\min}(C_0(t)) < \sigma/b$. In that case, the transformation (13) would not be well-defined $\forall t > 0$. Therefore, property 2 of section 3.1 would not be guaranteed for this transformation in the MIMO case.

3.4 Selective hysteresis + projection

To solve the problem of the pseudo-inverse $P_p(P_p^T P_p)^{-1}$ not being necessarily bounded in the MIMO case, an additional modification is brought to the parameter transformation. Essentially, the problem is that if the estimated matrix C_0 persists in being singular, despite the presence of excitation in certain (but obviously not all) channels, then the matrix C_{0c} must be updated selectively in the directions where there is excitation. The major challenge is to achieve this objective while maintaining the boundedness of C_{0c}^{-1} and the properties listed in section 3.1.

We consider the following transformation

$$\theta_c(t) = \begin{cases} \theta(t) & \text{if } T_{k-1} \leq t < t_k \quad \forall k > 0 \\ \theta(t) + P(t)V_P(t)\Sigma_P^{-1}(t)U_P^T(t)(C_{0f}^T(t)R_P^T(t) - C_0^T(t)) & \text{if } t_k \leq t < T_k \quad \forall k > 0 \end{cases} \quad (17)$$

where $V_P\Sigma_P^{-1}U_P^T$ is a lower rank approximation of $P_p(P_p^T P_p)^{-1}$. Specifically, we consider the following partitioned singular value decomposition of P_p^T

$$\begin{bmatrix} U_P & U_\epsilon \end{bmatrix}^T P_p^T \begin{bmatrix} V_P & V_\epsilon \end{bmatrix} = \begin{bmatrix} \Sigma_P & 0 \\ 0 & \Sigma_\epsilon \end{bmatrix} = \Sigma = \text{diag}\{\sigma_i\} \quad \sigma_i \geq \sigma_j \quad \forall i < j \quad (18)$$

such that

$$\begin{aligned} \Sigma_P &= \begin{bmatrix} \sigma_1 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \sigma_{N_P} \end{bmatrix} > (\sigma_{N_P+1} + \delta_P)I \geq (\epsilon_P + \delta_P)I \\ \Sigma_\epsilon &= \begin{bmatrix} \sigma_{N_P+1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \vdots & \ddots & \sigma_{p-1} & 0 \\ 0 & \dots & 0 & \sigma_p \end{bmatrix} \leq \begin{bmatrix} \epsilon_P + (p - N_P - 1)\delta_P & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \epsilon_P + \delta_P & 0 \\ \vdots & \ddots & 0 & \epsilon_P \end{bmatrix} \\ &\leq (\epsilon_P + (p - N_P - 1)\delta_P)I \end{aligned} \quad (19)$$

N_P is the size of Σ_P and, therefore, $U_P \in \mathbb{R}^{p \times N_P}$, $V_P \in \mathbb{R}^{2p \times N_P}$, $U_\epsilon \in \mathbb{R}^{p \times (p-N_P)}$, and $V_\epsilon \in \mathbb{R}^{2p \times (p-N_P)}$. The constants ϵ_P and δ_P are arbitrarily chosen, but must satisfy the following conditions

$$\begin{aligned} 0 &< \epsilon_P < (1 - 1/a) \frac{\sigma k_0}{(1 + (p-1)/\alpha_P)(\sigma_{\theta^*} + |\theta(0)|)} \quad \text{with} \quad \alpha_P > 1 \\ 0 &< \delta_P = \epsilon_P/\alpha_P \end{aligned} \quad (20)$$

where σ_{θ^*} is a known upper bound of $|\theta^*|$. Basically, Σ_ϵ contains the singular values of P_p^T no greater than ϵ_P and the ones which are sufficiently close to them (by increment δ_P). Σ_P contains the larger singular values such that $\sigma_{\min}(\Sigma_P) > \sigma_{\max}(\Sigma_\epsilon) + \delta_P$. The difference δ_P is found to be required to guarantee continuity in the proof.

Finally, $R_P^T = V_f U_0^T$, where V_f is an orthonormal basis of the subspace $\mathcal{X} = \mathcal{R}(C_{0_f} U_P U_P^T)$ of size N_P and U_0 is an orthonormal basis of the subspace $\mathcal{Y} = \mathcal{R}(C_0 U_\epsilon U_\epsilon^T)^\perp$ also of size N_P . Therefore, given the properties of projections (cf. Definition 2)

$$\begin{aligned} V_f^T V_f &= I \quad \text{and} \quad V_f V_f^T = P_{\mathcal{R}(C_{0_f} U_P U_P^T)} \triangleq P_{\mathcal{X}} \in \mathbb{R}^{p \times p} \\ U_0^T U_0 &= I \quad \text{and} \quad U_0 U_0^T = P_{\mathcal{R}(C_0 U_\epsilon U_\epsilon^T)^\perp} \triangleq P_{\mathcal{Y}} \in \mathbb{R}^{p \times p} \end{aligned} \quad (21)$$

If $\sigma_{\min}(P_p^T(t)) > \epsilon_P$ then $N_P = p$, and $P_{\mathcal{X}} = P_{\mathcal{Y}} = I$. If $\sigma_{\min}(P_p^T(t)) \leq \epsilon_P$ then $N_P < p$ and $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ are given by

$$\begin{aligned} P_{\mathcal{X}} &= C_{0_f} U_P (U_P^T C_{0_f}^T C_{0_f} U_P)^{-1} U_P^T C_{0_f}^T \\ P_{\mathcal{Y}} &= (I - C_0 U_\epsilon (U_\epsilon^T C_0^T C_0 U_\epsilon)^{-1} U_\epsilon^T C_0^T) \end{aligned} \quad (22)$$

Indeed, referring to Definition 2, we let $A = C_{0_f} U_P U_P^T$ and $U = [U_P \ U_\epsilon]$, so that $B = C_{0_f} U_P$ and $P_{\mathcal{X}}$ is given by (22). This assumes that $B = C_{0_f} U_P$ has full column rank, i.e., $U_P^T C_{0_f}^T C_{0_f} U_P$ nonsingular. However, since C_{0_f} is nonsingular, $(U_P^T C_{0_f}^T C_{0_f} U_P)^{-1}$ is well-defined and the number of columns of V_f is equal to N_P . A similar derivation applies for $P_{\mathcal{Y}}$. From (15), it can be checked that

$$U_\epsilon^T C_0^T(t) = U_\epsilon^T C_0^{*T} + U_\epsilon^T P_p^T(t) \frac{\phi(\tau_j)}{k_0} = U_\epsilon^T C_0^{*T} + \Sigma_\epsilon V_\epsilon^T \frac{\phi(\tau_j)}{k_0} \quad (23)$$

where τ_j is the last covariance resetting time instant, i.e. $\tau_j < t < \tau_{j+1}$. Therefore, given (19) and (20)

$$\sigma_{\min}(U_\epsilon^T C_0^T) \geq \sigma - (\epsilon_P + (p - N_P - 1)\delta_P) \frac{|\phi(0)|}{k_0} > \sigma/a \quad (24)$$

so that $(U_\epsilon^T C_0^T C_0 U_\epsilon)^{-1}$ is well-defined and the number of columns of U_0 is indeed equal to N_P .

Comments:

To guarantee some continuity properties and the uniqueness of V_f and U_0 , the Gram-Schmidt orthogonalization procedure with memory described in the appendix is applied to $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ to compute V_f and U_0 respectively. Using this procedure, we also have that if $\sigma_{\min}(P_p^T(t)) > \epsilon_P$ then $N_P = p$, $R_P^T(t) = I$, $V_P(t) \Sigma_P^{-1}(t) U_P^T(t) = P_p(t) (P_p^T(t) P_p(t))^{-1}$, and the parameter transformation (17) is simply identical to the parameter transformation (13).

It can also be shown that $N_P \geq 1$. Indeed, if $\exists t > 0$ such that $N_P = 0$, then from (24), $\sigma_{\min}(C_0) > \sigma/a$. Therefore, $\exists k > 0$ such that $T_{k-1} \leq t < t_k$ and $\theta_c(t) = \theta(t)$. In other words, if P_p was to be small in all directions, then it would mean that there had been enough excitation so that C_0 would be close to C_0^* and one would be out of the singularity region. Therefore, $N_P \geq 1$, $\sigma_{\min}(\Sigma_P) > \epsilon_P + \delta_P$ and the parameter transformation (17) is well-defined. Basically, the inverse $(P_p^T P_p)^{-1}$ has been replaced by a lower order inverse, whose existence and boundedness can be guaranteed.

Finally, it can be verified that $C_{0_c}^{-1} \in L_\infty$. Indeed, if $T_{k-1} \leq t < t_k$ or if $t_k \leq t < T_k$ and $\sigma_{\min}(P_p^T(t)) > \epsilon_P$, then $C_{0_c}^T(t) = C_{0_f}^T(t)$ whose inverse is always bounded. If $t_k \leq t < T_k$ and $\sigma_{\min}(P_p^T) \leq \epsilon_P$, then

$$\begin{aligned} C_{0_c}^T(t) &= C_0^T(t) + P_p^T(t)V_P(t)\Sigma_P^{-1}(t)U_P^T(t)(C_{0_f}^T(t)R_P^T(t) - C_0^T(t)) \\ &= C_0^T(t) + U_P(t)U_P^T(t)(C_{0_f}^T(t)R_P^T(t) - C_0^T(t)) \\ &= U_P(t)U_P^T(t)C_{0_f}^T(t)R_P^T(t) + U_\epsilon(t)U_\epsilon^T(t)C_0^T(t) \end{aligned} \quad (25)$$

Therefore, using (21) and (22),

$$\begin{aligned} C_{0_c}^T C_{0_c} &= U_P U_P^T C_{0_f}^T V_f V_f^T C_0 U_P U_P^T + U_\epsilon U_\epsilon^T C_0^T C_0 U_\epsilon U_\epsilon^T \\ &= \begin{bmatrix} U_P & U_\epsilon \end{bmatrix} \begin{bmatrix} U_P^T C_{0_f}^T C_0 U_P & 0 \\ 0 & U_\epsilon^T C_0^T C_0 U_\epsilon \end{bmatrix} \begin{bmatrix} U_P^T \\ U_\epsilon^T \end{bmatrix} \end{aligned} \quad (26)$$

so that $\sigma_{\min}(C_{0_c}) \geq \sigma/b$.

Finally, given (17) and all the previous definitions, the parameter transformation with hysteresis will have the following form

$$\theta_c(t) = \theta(t) + P(t)Q(t) \quad (27)$$

with

$$Q(t) = \begin{cases} 0 & \text{if } T_{k-1} \leq t < t_k \quad \forall k > 0 \\ P_p(t)(P_p^T(t)P_p(t))^{-1}(C_{0_f}^T(t) - C_0^T(t)) & \text{if } \begin{cases} t_k \leq t < T_k \quad \forall k > 0 \\ \text{and } \sigma_{\min}(P_p^T(t)) > \epsilon_P \end{cases} \\ V_P(t)\Sigma_P^{-1}(t)U_P^T(t)(C_{0_f}^T(t)R_P^T(t) - C_0^T(t)) & \text{if } \begin{cases} t_k \leq t < T_k \quad \forall k > 0 \\ \text{and } \sigma_{\min}(P_p^T(t)) \leq \epsilon_P \end{cases} \end{cases} \quad (28)$$

Lemma 2 : Properties of the hysteresis transformation

Assuming that $\psi \in L_{\infty c}$ and that the parameter estimation algorithm is defined by (9), then

1. The transformation is always uniquely defined.
2. $C_{0_c}^{-1} \in L_\infty$.
3. $Q \in L_\infty$, $\theta_c \in L_\infty$, and $\phi_c \in L_\infty$, where $\phi_c = \theta_c - \theta^*$.
4. $\beta_c = \frac{\phi_c^T \psi}{1 + \|\psi\|_\infty} \in L_2 \cap L_\infty$.
5. If $\lim_{t \rightarrow \infty} \phi(t) = 0$ then $\lim_{t \rightarrow \infty} \phi_c(t) = 0$.

6. $(t_k - T_{k-1})$ and $(T_k - t_k)$ are bounded below $\forall k$.
7. $\{T_k\}$ and $\{t_k\}$ are finite sets.
8. If $P(t)$ and $\phi(t)$ converge to some P_∞ and ϕ_∞ then $\phi_c(t)$ converges to some $\phi_{c\infty}$.

The proof is in de Mathelin (1993).

Comments:

The fact that $Q \in L_\infty$ is obtained because, even though $P^{-1}(t)$ is not necessarily bounded as t increases, Σ_P^{-1} exists and is always bounded when $Q \neq 0$. In other words, should P^{-1} be unbounded, either Σ_P^{-1} will continue to exist or the algorithm will come out of the hysteresis region. The fourth property is most important: it shows that the main property on β in the estimation algorithm remains valid when the estimated parameter error ϕ is replaced by the controller parameter error ϕ_c . The fact that the intervals $(t_k - T_{k-1})$ and $(T_k - t_k)$ are bounded below comes from the normalized nature of the adaptation algorithm and from the separation of the freezing and unfreezing levels a and b . It eliminates the possibility of having an infinite number of jumps in a finite interval of time. The seventh property tells us that there is a finite number of passages in the hysteresis loop. After a while, the algorithm will settle and no more jumps will occur. The property is the consequence of the convergence of ϕ (not necessarily to zero). The algorithm could actually settle inside the hysteresis loop. In that event, we will see in the following section that the stability properties of the adaptive algorithm are preserved. The last property is obtained because of the difference δ_P between Σ_P and Σ_c and because of the use of the particular Gram-Schmidt orthogonalization procedure defined in appendix.

Note that from (27) and (28) with (25),

$$\begin{aligned} U_P^T(t)C_{0c}^T(t) &= U_P^T(t)C_{0f}^T(t)R_P^T(t) \\ U_c^T(t)C_{0c}^T(t) &= U_c^T(t)C_0^T(t) = U_c^T(t)C_0^{*T}(t) + \Sigma_c(t)V_c^T(t) \frac{\phi(\tau_k)}{k_0} \end{aligned}$$

where τ_k is the last covariance resetting time instant. Basically, this particular parameter transformation unfreezes C_{0c} in the directions where $(C_0^T - C_0^{*T})$ is sufficiently small, using the knowledge that there has been sufficient excitation in those directions.

4 Stability

If we define the model signals, ψ_m , as the signals ψ when $\phi = 0$, then we can define the regressor error, e_ψ , as $e_\psi = \psi - \psi_m$.

Theorem 1 : Stability of the MRAC system

Consider the MIMO MRAC system with the parameter estimation algorithm and the hysteresis transformation described previously. If the reference input $r \in L_\infty$ and is piecewise continuous, then

- All states of the adaptive system are bounded functions of time.
- The output error $e_0 = y_p - y_m \in L_\infty$ and $\lim_{t \rightarrow \infty} e_0 = 0$.

- The regressor error $e_\psi = \psi - \psi_m \in L_\infty \cap L_2$ and $\lim_{t \rightarrow \infty} e_\psi = 0$.

The proof is an extension of the SISO proof of Sastry & Bodson (1989) and can be found in de Mathelin (1993).

5 Example

Let the plant be a (2×2) system of order 4, with the following transfer matrix

$$P(s) = \begin{bmatrix} \frac{s^2+s}{s^3+3s^2+3s+2} & \frac{-2s^3-7s^2-7s-4}{s^4+4s^3+6s^2+5s+2} \\ \frac{1}{s^3+3s^2+3s+2} & \frac{s^3+3s^2+3s+1}{s^4+4s^3+6s^2+5s+2} \end{bmatrix}$$

Let $\Lambda(s) = \text{diag}\{(s + \lambda)\}$, $L(s) = \text{diag}\{(\frac{s}{l} + 1)\}$, and $M(s) = H(s) = \text{diag}\{\frac{1}{(s+1)}\}$. Then, $C_0^* = K_p^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ and $\sigma_{\min}(C_0^*) = 0.4142$. We present simulations of the adaptive system with and without the hysteresis transformation to show how important the parameter transformation algorithm is for convergence and stability. In our simulations, the different parameters are set to the following values: $\lambda = 4$, $l = 10$, $g = 10$, $\gamma = 0$, $k_0 = 1$, $k_1 = 0$, $k_2 = 0$, $\sigma = 0.1$, $a = 2$, $b = 4$, $\epsilon_P = 10^{-6}$, $\alpha_P = 2$, and the initial estimate of the parameters $\theta^T(0) = \begin{bmatrix} -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.5 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Therefore, $\sigma_{\min}(C_0(0)) = 0.5 > \sigma$.

The reference inputs $r_1 = \sin(5t) + \sin(7t) + \sin(10t)$ and $r_2 = \sin(6t) + \sin(8t) + \sin(9t)$ contain 6 frequency components each and, consequently, are sufficiently rich to guarantee persistency of excitation (cf. de Mathelin & Bodson (1992)).

In the first simulation, the adaptive control algorithm has been implemented with the hysteresis transformation. The evolution of the norm of the output error, $e_0 = y_p - y_m$, is shown in Fig. 2. After a transient of about 20 seconds, the output error rapidly converges to zero. The singular values $\sigma_{\min}(C_{0e})$ and $\sigma_{\min}(C_0)$ are compared in Fig. 3 and Fig. 4 for, respectively, the first 5 and 20 seconds of the simulation. One sees that the system goes three times into the hysteresis before finally settling outside the hysteresis region. The time instants (in seconds) for entering and leaving the hysteresis are $t_1 = 0.165$, $T_1 = 0.208$, $t_2 = 0.509$, $T_2 = 1.762$, $t_3 = 2.224$, and $T_3 = 16.319$. It can also be seen that $\sigma_{\min}(C_0)$ goes to zero 6 times during the simulation before converging toward $\sigma_{\min}(C_0^*) = 0.4142$ (see Fig. 5). During the simulation, $\sigma_{\min}(P_p^T)$ stayed above ϵ_P inside the hysteresis region.

Finally, the same simulation was executed without the parameter transformation algorithm. The norm of the output error is shown in Fig. 6 and $\sigma_{\min}(C_0)$ in Fig. 7. Clearly, after a first quick passage through the zero region, $\sigma_{\min}(C_0)$ converges into the zero region and does not leave. The control parameter matrix $C_{0e} = C_0$ settles in a singularity region and, consequently, suppresses part of the reference input excitation. Model matching is now impossible and, as Fig. 6 clearly shows, the output error does not converge to zero.

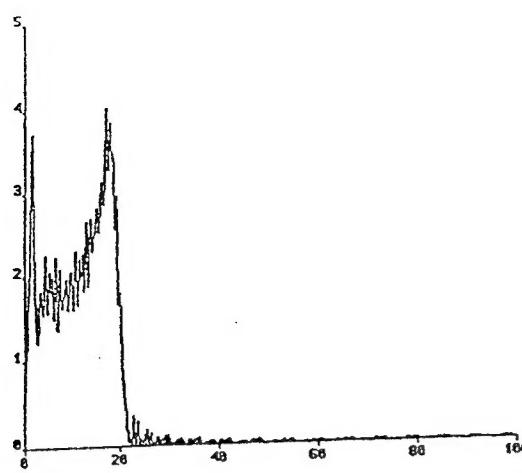


Figure 2: $|e_0(t)|$

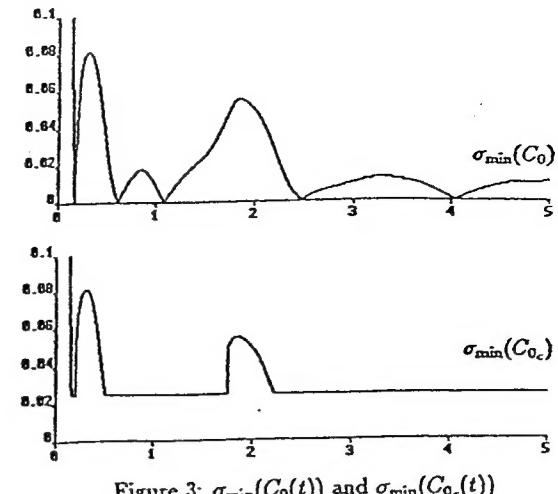


Figure 3: $\sigma_{\min}(C_0(t))$ and $\sigma_{\min}(C_{0_c}(t))$

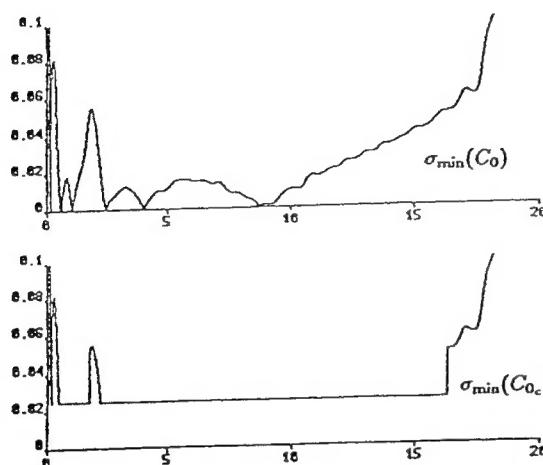


Figure 4: $\sigma_{\min}(C_0(t))$ and $\sigma_{\min}(C_{0_c}(t))$

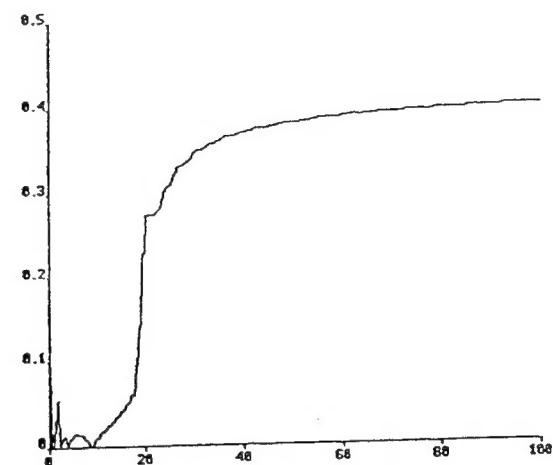


Figure 5: $\sigma_{\min}(C_0(t))$

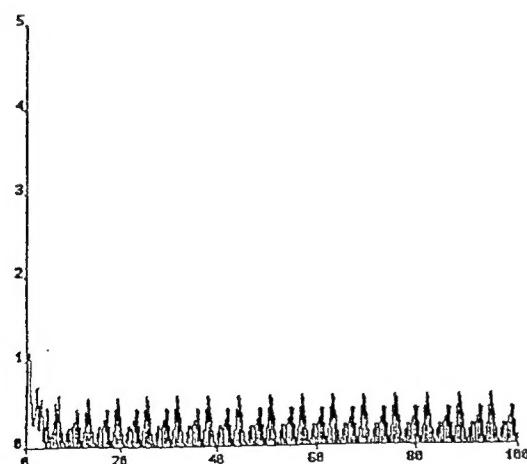


Figure 6: $|e_0(t)|$ (without parameter transformation)

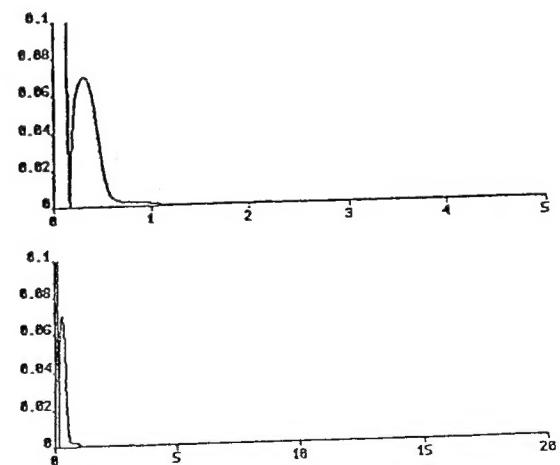


Figure 7: $\sigma_{\min}(C_0(t))$ (without parameter transformation)

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Appendix

Gram-Schmidt orthogonalization with memory

Let $W(t) \in \mathbb{R}^{p \times p}$ be a matrix of rank r function of time and let $h > 0$ be a constant to be fixed later.

1. At time $t = 0$, apply the following procedure:

$$\begin{aligned}
 Y_1 &= W_{k^{(1)}} \quad \text{where} \quad W_{k^{(1)}} = k^{(1)}\text{th column of } W \text{ s.t.} \\
 &\quad |W_{k^{(1)}}| = \max_k |W_k| \text{ and } k^{(1)} \text{ minimum} \\
 W^{(1)} &= (I - \frac{Y_1 Y_1^T}{Y_1^T Y_1}) W \quad (W_{k^{(1)}}^{(1)} = 0) \\
 Y_2 &= W_{k^{(2)}}^{(1)} \quad \text{where} \quad W_{k^{(2)}}^{(1)} = k^{(2)}\text{th column of } W^{(1)} \text{ s.t.} \\
 &\quad |W_{k^{(2)}}^{(1)}| = \max_k |W_k^{(1)}| \text{ and } k^{(2)} \text{ minimum} \\
 W^{(2)} &= (I - \frac{Y_2 Y_2^T}{Y_2^T Y_2}) W^{(1)} \quad (W_{k^{(1)}}^{(2)} = W_{k^{(2)}}^{(2)} = 0) \\
 &\vdots \\
 Y_r &= W_{k^{(r)}}^{(r-1)} \quad \text{where} \quad W_{k^{(r)}}^{(r-1)} = k^{(r)}\text{th column of } W^{(r-1)} \text{ s.t.} \\
 &\quad |W_{k^{(r)}}^{(r-1)}| = \max_k |W_k^{(r-1)}| \text{ and } k^{(r)} \text{ minimum} \\
 W^{(r)} &= (I - \frac{Y_r Y_r^T}{Y_r^T Y_r}) W^{(r-1)} = 0
 \end{aligned}$$

Then, the matrix

$$X(0) = \left[\begin{array}{ccc} \frac{Y_1}{|Y_1|} & \dots & \frac{Y_r}{|Y_r|} \end{array} \right]$$

is an orthogonal basis of $\mathcal{R}(W(0))$, the space of dimension r spanned by the columns of $W(0)$. The order of selection of the columns of $W(0)$, $\{k^{(i)}\}_{i=0}$, $i = 1, \dots, r$ is uniquely defined by this procedure. Therefore, the matrix $X(0)$ is also uniquely defined.

2. Keep the initial order of selection of the columns, $\{k^{(i)}\}_{i=0}$, for the orthogonalization of $W(t)$, $t > 0$, until

$$\exists t = t_1(W) > 0 \quad \exists 1 \leq j \leq r \quad \exists 1 \leq l \leq p \quad \text{s.t.} \quad |W_{k^{(j)}}^{(j-1)}(t)|^2 + h < |W_l^{(j-1)}(t)|^2$$

where $W^{(0)} = W$. Then, at $t = t_1(W)$ the procedure defined for $t = 0$ is applied for the computation of $X(t_1)$, and a new order of selection of the columns, $\{k^{(i)}\}_{i=t_1(W)}$, is found.

3. Finally, continue this procedure for the orthogonalization of $W(t)$, $t > t_1(W)$. The set $\{t_k(W)\}$ are the time instants when the order of selection of the columns of W is changed. The matrix $X(t)$ will be uniquely defined for all $t \geq 0$.

The procedure is applied with $W = P_X$, leading to $X = V_f$, and similarly to P_Y , leading to U_0 . The advantage of using this Gram-Schmidt orthogonalization with memory is that the matrices V_f and U_0 are uniquely defined for given matrices P_X and P_Y . The constant h is chosen sufficiently small that no vector Y_i may become equal to zero, while preventing the order of selection of the columns to change an infinite number of time during a finite time

interval. Note that using the Gram-Schmidt orthogonalization procedure with memory for $P_x = I$ or $P_y = I$ leads to $V_f = I$ and $U_0 = I$. Finally, it can be shown (cf. de Mathelin (1993)) that if $h \leq \frac{1}{2p}$, then the Gram-Schmidt orthogonalization with memory is always well-defined for V_f and U_0 . In other words, $|Y_i(t)| > 0 \forall t \geq 0, \forall i$.